Symmetric Prismatic Tensegrity Structures: Part II. Symmetry-adapted Formulations

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Abstract

This paper presents analytical formulations for the symmetry-adapted equilibrium, force density and geometrical stiffness matrices for prismatic tensegrity structures with dihedral symmetries. An analytical expression for the infinitesimal mechanisms is also derived.

Key words: Tensegrity; Symmetry; Dihedral Group; Group Representation Theory; Block-diagonalisation.

1. Introduction

Analytical methods can provide an in-depth understanding of a whole class of structures that is not afforded by numerical solutions. In the first part of our study (Zhang *et al.*, 2008a), the (prestress) stability of the symmetric prismatic tensegrity structures has been investigated to show that connectivity of horizontal and vertical cables and the ratio of height to radius are critical to stability, using analytical symmetry-adapted (block-diagonal) matrices. This paper presents the analytical formulations of these matrices in a direct manner, based on group representation theory.

Stability, prestress stability and super stability are three stability criteria used for tensegrity structures. Among these, *stability* indicates positive definiteness of the tangent stiffness matrix $\mathbf{K}$, which, as in Part I, or Guest (2006), can be written as

$$\mathbf{K} = \mathbf{A}\hat{\mathbf{G}}\mathbf{A}^{\mathrm{T}} + \mathbf{S}, \tag{1}$$

where $\mathbf{A}$ is the equilibrium matrix, $\mathbf{S}$ is the geometrical stiffness matrix, and $\hat{\mathbf{G}}$ is a diagonal matrix containing modified axial stiffnesses of members. *Prestress stability* is a simplified version of stability, where members are assumed to be rigid; equivalently they have infinite stiffness. A structure is said to be prestress stable when the reduced stiffness

matrix $\mathbf{Q}$, defined as the quadratic form of the geometrical stiffness matrix $\mathbf{S}$ with respect to the mechanisms $\mathbf{M}$, is positive definite:

$$\mathbf{Q} = \mathbf{M}^{\mathrm{T}}\mathbf{S}\mathbf{M}, \tag{2}$$

where columns of $\mathbf{M}$ are first-order mechanisms of the structure lying in the null-space of $\mathbf{A}^{\mathrm{T}}$. *Super stability* is superior to stability and prestress stability—a super stable structure is (prestress) stable, and furthermore, any 'stretched' version of it is also super stable. Connelly (1999), and Zhang and Ohsaki (2007) have discussed the conditions for super stability of a tensegrity structure: positive semi-definiteness of the geometrical stiffness matrix $\mathbf{S}$ is a necessary condition.

For a structure with m nodes in d-dimensions, $\mathbf{S}$ is an md-by-md matrix. It can be written in terms of a simpler m-by-m force density matrix $\mathbf{E}$, using the Kronecher product ($\otimes$) with a d-by-d identity matrix $\underset{d \times d}{\mathbf{I}}$:

$$\mathbf{S} = \underset{d \times d}{\mathbf{I}} \otimes \mathbf{E}, \tag{3}$$

if the coordinates are written as $[\ldots, x_i, x_{i+1}, \ldots, y_i, y_{i+1}, \ldots, z_i, z_{i+1}, \ldots]^{\mathrm{T}}$. Thus, for each eigenvalue of $\mathbf{E}$, there will be d copies of that eigenvalue for $\mathbf{S}$, and the positive definiteness of $\mathbf{S}$ can be verified by that of $\mathbf{E}$.

The positive definiteness of a matrix can be verified by investigating the sign of its smallest eigenvalue: the matrix is positive definite if its smallest eigenvalue is positive, and is positive semi-definite if it is equal to zero. Since the size of the stiffness matrix increases in proportion to the number of nodes, eigenvalue analysis of the matrices will be more time consuming when the structure becomes more complex. However, for symmetric structures, there is a systematic way to reduce the computational cost by transforming the current (internal and external) coordinate systems into symmetry-adapted systems. In this way, the matrices are rewritten in a symmetry-adapted, block-diagonal form: independent blocks are located on the leading diagonal. As the eigenvalues of a matrix are not changed by a transformation of coordinate system, the positive definiteness of a matrix can be verified by considering the eigenvalues of its independent blocks, which are of smaller dimension than the original matrix.

There are a number of methods to block-diagonalise the matrices for symmetric structures (Kangwai *et al.*, 1999): in these methods, the symmetry-adapted matrices are derived in a numerical manner, using transformation matrices. However, one numerical transformation only deals with one specific structure. In this paper, we present a direct strategy for deriving symmetry-adapted matrices in an analytical way for prismatic tensegrity structures with dihedral symmetry, and demonstrate that these formulations can deal with a whole class of structures. These symmetry-adapted formulations were used to investigate the (prestress) stability of prismatic tensegrity structures in the first part of this study (Zhang *et al.*, 2008a), and to present the super stability condition for dihedral 'star' tensegrity structures (Zhang *et al.*, 2008b).

Following this section, the paper is organised as follows. Section 2 gives a brief introduction to group representation theory, for the dihedral group in particular. Section 3 and Section 4 respectively formulate the symmetry-adapted force density matrix and
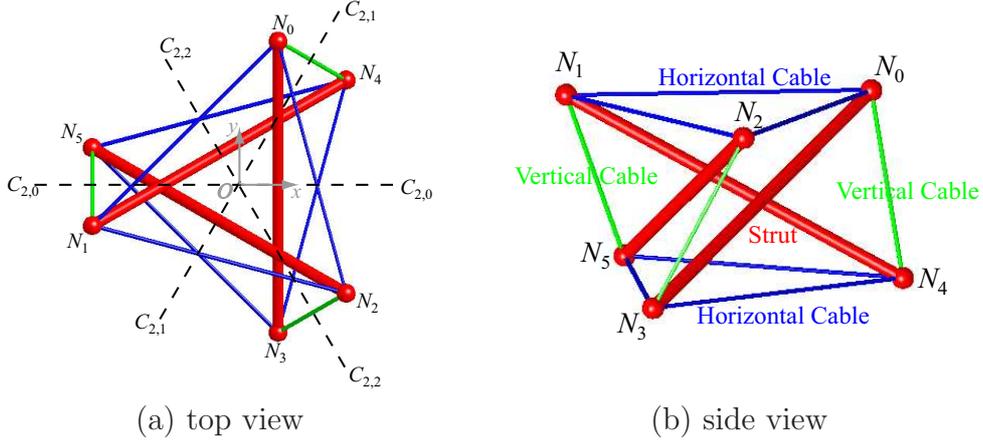
(a) top view   (b) side view

Fig. 1. The simplest three-dimensional tensegrity, a prismatic tensegrity structure with $\mathbf{D}_3$ dihedral symmetry. The group $\mathbf{D}_3$ has six symmetry operations: the identity, rotation by $2\pi/3$ and $4\pi/3$ about the z-axis, and rotation by $\pi/2$ about the three axes $C_{2,0}$, $C_{2,0}$, $C_{2,2}$. The structure consists of six nodes and six horizontal cables having a one-to-one correspondence with symmetry operations, and three vertical cables and struts having one-to-two correspondence with symmetry operations.

geometrical stiffness matrix; the symmetry-adapted force density matrix is used to obtain conditions for self-equilibrium and super stability of prismatic tensegrity structures. The symmetry-adapted equilibrium matrix is presented in Section 5, using the concept of unitary member direction; and symmetry-adapted mechanisms are then derived from its transpose. Section 6 briefly discusses and concludes the study.

2. Group and Matrix Representation

Symmetry of a structure can be systematically dealt with by group representation theory. To prepare for the symmetry-adapted formulations in the coming sections, some basic concepts of group and its matrix representation are briefly introduced in this section.

2.1 Group

A *group* is defined by a set of elements and the combination rules between these elements, satisfying the following four general criteria (Kettle, 1995; Bishop, 1973):

(1) *Closure*: any two elements of the group must combine to give an element that is also a member of the group.
(2) *Associativity*: the associative law of combination must be satisfied.
(3) *Identity*: the group must contain an element, called *identity element*, that commutes with all the other elements and also leaves them unchanged.
(4) *Inverse*: the inverse of every element in the group is also a member of the group.

The *order* of a group denotes the number of elements in it. In a description of the symmetry properties of a structure, the elements are called *symmetry operations*. A symmetry operation is an operation which moves the structure in such a way that its final position is physically indistinguishable from its initial position. If there is at least one point (called the central point) in the structure that does not change its position by any symmetry operations of a group, that group is called *point group*. There are five different

Table 1
Irreducible matrix representations $\mathbf{R}_i^\mu$ of the dihedral group $\mathbf{D}_n$. Each row corresponds to a representation μ of the group. C_{ik} and S_{ik} respectively denote $\cos(2ik\pi/n)$ and $\sin(2ik\pi/n)$. x, y, z and R_x, R_y, R_z respectively stand for symmetry operations of the corresponding coordinates and rotations about those axes.

μ	C_n^i	$C_{2,i}$		notes
A_1	1	1		
A_2	1	-1	$z,\ R_z$	
(B_1)	$(-1)^i$	$(-1)^i$		n even
(B_2)	$(-1)^i$	$(-1)^{(i+1)}$		n even
E_1	$\begin{bmatrix} C_i & -S_i \\ S_i & C_i \end{bmatrix}$	$\begin{bmatrix} C_i & S_i \\ S_i & -C_i \end{bmatrix}$	$(x,y)\ (R_x, R_y)$	
E_k	$\begin{bmatrix} C_{ik} & -S_{ik} \\ S_{ik} & C_{ik} \end{bmatrix}$	$\begin{bmatrix} C_{ik} & S_{ik} \\ S_{ik} & -C_{ik} \end{bmatrix}$		$k \in \{2, \ldots, p\}$
	$\mathbf{R}_i^\mu$	$\mathbf{R}_{n+i}^\mu$		

types of symmetry operation in a point group: (1) identity, (2) rotation about an axis, (3) reflection in a plane, (4) improper rotation (rotation about an axis followed by reflection in a plane perpendicular to the rotation axis, (5) inversion (reflection through a central point; equivalently an improper rotation by π about any axis).

The prismatic tensegrity structures that are of interest in this study, such as the structure shown in Fig. 1, have dihedral symmetry: they are unchanged by any of the symmetry operations in the dihedral group $\mathbf{D}_n$. The dihedral group is a point group, and consists of: (1) the identity operation $E(\equiv C_n^0)$; (2) $n-1$ rotation operations C_n^i ($i \in \{1, \ldots, n-1\}$) about a principal axis; and (3) n two-fold rotation operations $C_{2,i}$ ($i \in \{0, \ldots, n-1\}$) about axes perpendicular to the principal axis. The order of a dihedral group $\mathbf{D}_n$ is $2n$. For convenience, we take the z-axis of the Cartesian coordinate system as the principal axis, and take the origin as the central point of the group.

A prismatic tensegrity structure with $\mathbf{D}_n$ symmetry consists of $2n$ nodes, $2n$ horizontal cables, n vertical cables and n struts. We assign that cables carry tension and struts carry compression. Nodes of a prismatic structure lie in two parallel planes; horizontal cables connect the nodes in the same plane, and vertical cables and struts connect those in different planes. The nodes and horizontal cables have one-to-one correspondence to the symmetry operations of the group, while the struts and vertical cables have one-to-two correspondence (Zhang *et al.*, 2008a).

2.2 Matrix Representation

The *group multiplication table* describes combinations of two operations (elements) of a group. If a set of matrices obeys the group multiplication table of a group, these matrices are said to form a *matrix representation* of that group. A matrix representation that can be reduced to a linear combination (direct sum) of several matrix representations is called *reducible matrix representation*, otherwise, they form an *irreducible matrix representation*. *Characters* are defined as the traces of irreducible representation matrices. They will be shown to be important in identifying the structures of symmetry-adapted matrices. The characters of the irreducible representations for point symmetry groups can be found in books of character tables, e.g., Altmann and Herzig (1994).

A dihedral group $\mathbf{D}_n$ has two or four one-dimensional irreducible matrix representations:

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for n odd, they are A_1 and A_2, and for n even, they are A_1, A_2, B_1 and B_2. For $n > 2$, there are also p two-dimensional irreducible matrix representations E_k ($k = 1, \ldots, p$) where

$$p = \begin{cases} (n-1)/2, & n \text{ odd} \\ (n-2)/2, & n \text{ even} \end{cases}. \tag{4}$$

The irreducible matrix representations of a dihedral group $\mathbf{D}_n$ are listed in Table 1. The one-dimensional matrix representations are unique, and their characters are the representation matrices themselves; characters of the two-dimensional representations are also unique—$2C_{ik}$ for the cyclic rotation C_n^i of E_k and zero for the two-fold rotation $C_{2,i}$, but there is some limited choice for the representation matrices. The symbols x, y and z in the fourth column of Table 1 respectively stand for x-, y- and z-coordinates, and R_x, R_y and R_z stand for rotations about these axes (Atkins *et al.*, 1970). We will show in Section 3 that the blocks of the symmetry-adapted force density matrix corresponding to the representations that stand for coordinates—A_2 and E_1 representations in the case of dihedral group—should be singular to ensure a non-degenerate configuration.

3. Symmetry-adapted Force Density Matrix

This section presents a direct strategy to find the symmetry-adapted force density matrix: sub-blocks of the matrix are written as sums of the products of force densities and the corresponding irreducible representation matrices. This analytical formulation significantly simplifies the self-equilibrium analysis and makes it possible to obtain super stability conditions (equivalent to those found by Connelly and Terrell (1995)) for the whole class of prismatic tensegrity structures.

3.1 Force Density Matrix

Every node of a prismatic tensegrity structure is connected by three different types of members: two horizontal cables, one vertical cable and one strut; and each type of members has the same self-stress and length. The nodes in the top plane of the structure are numbered from 0 to $n-1$, and those in the bottom are n to $2n-1$. Node N_0 in the top plane is connected to nodes N_h and N_{n-h} by horizontal cables, to node N_n by a strut and to N_{n+v} by a vertical cable: the parameters h and v are respectively used to describe connectivity of the horizontal and vertical cables of the structure. The members connected to other nodes can be determined by applying symmetry operations. We label a prismatic structure with $\mathbf{D}_n$ symmetry and connectivity of h and v as $\mathbf{D}_n^{h,v}$: for example, the structure in Fig. 1 is denoted as $\mathbf{D}_3^{1,1}$.

Let q_h, q_v and q_s denote the force densities (self-stress to length ratios) of horizontal cables, vertical cables and struts, respectively. Let $\mathcal{I}$ denote the set of members connected to node i. The (i,j)-component $E_{(i,j)}$ of the force density matrix $\mathbf{E} \in \Re^{2n \times 2n}$ is given as

$$E_{(i,j)} = \begin{cases} \sum\limits_{k \in \mathcal{I}} q_k & \text{for } i = j, \\ -q_k & \text{if nodes } i \text{ and } j \text{ are connected by member } k, \\ 0 & \text{for other cases.} \end{cases} \tag{5}$$

Denoting $q = 2q_h + q_s + q_v$, $\mathbf{E}$ can be written as follows from the numbering and

connectivity of nodes

$$\mathbf{E} = q\mathbf{R}_0 - q_h\mathbf{R}_h - q_h\mathbf{R}_{n-h} - q_s\mathbf{R}_n - q_v\mathbf{R}_{n+v}. \tag{6}$$

Define $\bar{\mathbf{I}}$ as an n-by-n matrix, of which the $(j+1)$-th entry in the j-th row is one while other entries in that row are zero. Thus, the matrices $\mathbf{R}_i$ ($i \in \{0, h, n-h, n, n+v\}$) are given by

$$\mathbf{R}_i = \begin{bmatrix} (\bar{\mathbf{I}}^i)^\mathrm{T} & \mathbf{O} \\ \mathbf{O} & \bar{\mathbf{I}}^i \end{bmatrix}, \qquad \text{for } 0 \le i < n;$$

$$\mathbf{R}_i = \begin{bmatrix} \mathbf{O} & \bar{\mathbf{I}}^{i-n} \\ (\bar{\mathbf{I}}^{i-n})^\mathrm{T} & \mathbf{O} \end{bmatrix}, \quad \text{for } n \le i < 2n. \tag{7}$$

where $\mathbf{O}$ is an n-by-n null matrix; i in $\bar{\mathbf{I}}^i$ is the power number, thus $\bar{\mathbf{I}}^0$ is the n-by-n identity matrix.

Consider for example, the simplest three-dimensional prismatic tensegrity structure $\mathbf{D}_3^{1,1}$ as shown in Fig. 1. The matrix $\bar{\mathbf{I}}$ is

$$\bar{\mathbf{I}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

and the force density matrix $\mathbf{E}$ is

$$
\mathbf{E} = q\left[\begin{array}{ccc|ccc}
1 & 0 & 0 & & & \\
0 & 1 & 0 & & \mathbf{O} & \\
0 & 0 & 1 & & & \\
\hline
& & & 1 & 0 & 0 \\
& \mathbf{O} & & 0 & 1 & 0 \\
& & & 0 & 0 & 1
\end{array}\right]
- q_h\left[\begin{array}{ccc|ccc}
0 & 0 & 1 & & & \\
1 & 0 & 0 & & \mathbf{O} & \\
0 & 1 & 0 & & & \\
\hline
& & & 0 & 1 & 0 \\
& \mathbf{O} & & 0 & 0 & 1 \\
& & & 1 & 0 & 0
\end{array}\right]
- q_h\left[\begin{array}{ccc|ccc}
0 & 1 & 0 & & & \\
0 & 0 & 1 & & \mathbf{O} & \\
1 & 0 & 0 & & & \\
\hline
& & & 0 & 0 & 1 \\
& \mathbf{O} & & 1 & 0 & 0 \\
& & & 0 & 1 & 0
\end{array}\right]
$$

$$
- q_s\left[\begin{array}{ccc|ccc}
& & & 1 & 0 & 0 \\
& \mathbf{O} & & 0 & 1 & 0 \\
& & & 0 & 0 & 1 \\
\hline
1 & 0 & 0 & & & \\
0 & 1 & 0 & & \mathbf{O} & \\
0 & 0 & 1 & & &
\end{array}\right]
- q_v\left[\begin{array}{ccc|ccc}
& & & 0 & 1 & 0 \\
& \mathbf{O} & & 0 & 0 & 1 \\
& & & 1 & 0 & 0 \\
\hline
0 & 0 & 1 & & & \\
1 & 0 & 0 & & \mathbf{O} & \\
0 & 1 & 0 & & &
\end{array}\right]
$$

$$
= \left[\begin{array}{ccc|ccc}
q & -q_h & -q_h & -q_s & -q_v & 0 \\
-q_h & q & -q_h & 0 & -q_s & -q_v \\
-q_h & -q_h & q & -q_v & 0 & -q_s \\
\hline
-q_s & 0 & -q_v & q & -q_h & -q_h \\
-q_v & -q_s & 0 & -q_h & q & -q_h \\
0 & -q_v & -q_s & -q_h & -q_h & q
\end{array}\right]
$$

3.2 Symmetry-adapted Formulation

Table 2
Traces of the reducible representation matrices $\mathbf{R}_i$, and characters of the irreducible representation matrices of dihedral group $\mathbf{D}_n$.

i	0	1	j	$n-1$	n	$n+1$	$n+j$	$2n-1$
$\mathrm{trace}(\mathbf{R}_i)$	$2n$	0	0	0	0	0	0	0
$\mu\backslash$operation	E	C_n^1	C_n^j	C_n^{n-1}	$C_{2,0}$	$C_{2,1}$	$C_{2,j}$	$C_{2,n-1}$
A_1	1	1	1	1	1	1	1	1
A_2	1	1	1	1	-1	-1	-1	-1
(B_1)	1	-1	$(-1)^j$	$(-1)^n$	1	-1	$(-1)^j$	$(-1)^n$
(B_2)	1	-1	$(-1)^j$	$(-1)^n$	1	-1	$(-1)^{j+1}$	$(-1)^{n+1}$
E_k	1	C_k	C_{jk}	$C_{(n-1)k}$	0	0	0	0

In this subsection, we present the direct formulation of the symmetry-adapted force density matrix.

Using $\mathbf{R}_1$ and $\mathbf{R}_n$ as generators, $\mathbf{R}_i$ can be written as

$$\mathbf{R}_i = \mathbf{R}_{j+n\tau} = (\mathbf{R}_1)^j(\mathbf{R}_n)^\tau, \quad \text{with } 0 \le j < n \ \text{ and } \ \tau \in \{0,1\}. \tag{8}$$

Thus, the matrices $\mathbf{R}_i$ $(i = 0,\ldots,2n-1)$ form a regular matrix representation of the dihedral group $\mathbf{D}_n$, since their products obey the multiplication table of $\mathbf{D}_n$. Moreover, they are reducible and can be rewritten as direct sum of the irreducible representation matrices: their traces indicate how many copies of each irreducible representation matrices are involved (Kettle, 1995).

Traces of the reducible representation matrices $\mathbf{R}_i$ corresponding to each symmetry operation are listed in Table 2 and summarised in $\Gamma(E)$:

$$\Gamma(E) = \{2n, 0, \ldots, 0; \ 0, \ldots, 0\}. \tag{9}$$

From characters of the irreducible matrices of dihedral group listed in Table 2, the reducible representation of the nodes can be written as a linear combination $\Gamma(E)$ of the irreducible representations in a general form as follows

$$\Gamma(E) = A_1 + A_2 + (B_1 + B_2) + 2\sum_{k=1}^{p} E_k. \tag{10}$$

We use $\tilde{(\cdot)}$ to denote the symmetry-adapted form of a matrix. $\Gamma(E)$ characterises the structure of the symmetry-adapted force density matrix $\tilde{\mathbf{E}}$:

(1) The number of the representation μ in $\Gamma(E)$ indicates dimensions of $\tilde{\mathbf{E}}^\mu$. Hence, the blocks corresponding to the one-dimensional representations are 1-by-1 matrices, and those of two-dimensional representations are 2-by-2 matrices.

(2) The dimensions of a representation indicate the number of times its corresponding block appears in the symmetry-adapted form; thus, each one-dimensional representation has only one copy, and each two-dimensional representation has two copies of blocks all lying along the leading diagonal of $\tilde{\mathbf{E}}$.

In summary, the structure of $\tilde{\mathbf{E}}$ can be written in a general form as follows

Table 3
Selected irreducible representation matrices corresponding to the nodes connecting to node 0.

μ	identity $\mathbf{R}_0^\mu$	horizontal cable $\mathbf{R}_h^\mu$	$\mathbf{R}_{n-h}^\mu$	strut $\mathbf{R}_n^\mu$	vertical $\mathbf{R}_{n+v}^\mu$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
B_1	1	$(-1)^h$	$(-1)^{n-h}$	1	$(-1)^v$
B_2	1	$(-1)^h$	$(-1)^{n-h}$	-1	$(-1)^{v+1}$
E_k	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} C_{hk} & -S_{hk} \\ S_{hk} & C_{hk} \end{bmatrix}$	$\begin{bmatrix} C_{hk} & S_{hk} \\ -S_{hk} & C_{hk} \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} C_{vk} & S_{vk} \\ S_{vk} & -C_{vk} \end{bmatrix}$

$$
\underset{2n\times 2n}{\tilde{\mathbf{E}}} =
\begin{bmatrix}
\underset{1\times 1}{\tilde{\mathbf{E}}^{A_1}} & & & & & & & & \\
& \underset{1\times 1}{\tilde{\mathbf{E}}^{A_2}} & & & & & \mathbf{O} & & \\
& & (\underset{1\times 1}{\tilde{\mathbf{E}}^{B_1}}) & & & & & & \\
& & & (\underset{1\times 1}{\tilde{\mathbf{E}}^{B_2}}) & & & & & \\
& & & & \underset{2\times 2}{\tilde{\mathbf{E}}^{E_1}} & & & & \\
& & & & & \underset{2\times 2}{\tilde{\mathbf{E}}^{E_1}} & & & \\
& & \mathbf{O} & & & & \ddots & & \\
& & & & & & & \underset{2\times 2}{\tilde{\mathbf{E}}^{E_p}} & \\
& & & & & & & & \underset{2\times 2}{\tilde{\mathbf{E}}^{E_p}}
\end{bmatrix}
$$

$$
= \tilde{\mathbf{E}}^{A_1} \oplus \tilde{\mathbf{E}}^{A_2}(\oplus\tilde{\mathbf{E}}^{B_1} \oplus \tilde{\mathbf{E}}^{B_2}) \oplus 2\sum_{k=1}^{p} \tilde{\mathbf{E}}^{E_k} \tag{11}
$$

where the blocks $\tilde{\mathbf{E}}^{B_1}$ and $\tilde{\mathbf{E}}^{B_2}$ corresponding to representations B_1 and B_2 exist only if n is even.

In conventional methods, $\tilde{\mathbf{E}}$ is obtained using the unitary transformation matrix $\mathbf{T} \in \Re^{2n\times 2n}$:

$$
\tilde{\mathbf{E}} = \mathbf{T}\mathbf{E}\mathbf{T}^{\mathrm{T}}, \tag{12}
$$

where $\mathbf{T}\mathbf{T}^{\mathrm{T}}$ is an identity matrix:

$$
\mathbf{T}\mathbf{T}^{\mathrm{T}} = \underset{2n\times 2n}{\mathbf{I}}. \tag{13}
$$

Rather than forming the transformation matrices $\mathbf{T}$, the block diagonal form of $\tilde{\mathbf{E}}$ can be found directly using Lemma 1 below.

Lemma 1 *The block $\tilde{\mathbf{E}}^\mu$ corresponding to representation μ of the symmetry-adapted force density matrix $\tilde{\mathbf{E}}$ can be written in a general form as*

$$
\tilde{\mathbf{E}}^\mu = q\mathbf{R}_0^\mu - q_h\mathbf{R}_h^\mu - q_h\mathbf{R}_{n-h}^\mu - q_s\mathbf{R}_n^\mu - q_v\mathbf{R}_{n+v}^\mu. \tag{14}
$$

Proof. Since the matrices $\mathbf{R}_i$ $(i = 0, \ldots, 2n - 1)$ defined in Eq. (7) or (8) form a regular matrix representation of the dihedral group $\mathbf{D}_n$, they can be written in the symmetry-

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adapted form as follows from Eq. (10) (Kettle, 1995)

$$\mathbf{R}_i = \mathbf{R}_i^{A_1} \oplus \mathbf{R}_i^{A_2} (\oplus \mathbf{R}_i^{B_1} \oplus \mathbf{R}_i^{B_2}) \oplus 2 \sum_{k=1}^{p} \mathbf{R}_i^{E_k}.$$

Hence, the force density matrix $\mathbf{E}$ can be written in a block-diagonal form as follows from its definition in Eq. (6)

$$\tilde{\mathbf{E}}^{\mu} = q\mathbf{R}_0^{\mu} - q_h \mathbf{R}_h^{\mu} - q_h \mathbf{R}_{n-h}^{\mu} - q_s \mathbf{R}_n^{\mu} - q_v \mathbf{R}_{n+v}^{\mu},$$

which proves the lemma. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ ▱

3.3 Self-equilibrated Configuration and Super Stability

To ensure a non-degenerate tensegrity structure in three-dimensional space, the force density matrix $\mathbf{E}$ (or equivalently $\tilde{\mathbf{E}}$) should have rank deficiency of at least four (Connelly, 1982; Zhang and Ohsaki, 2006). Rank deficiency of a symmetric matrix can be calculated by finding the number of zero eigenvalues, and we should do this on a block by block basis for $\tilde{\mathbf{E}}$.

From Eq. (14), the block $\tilde{\mathbf{E}}^{A_1}$ is always equal to zero, as all representation matrices $\mathbf{R}_i^{A_1}$ are equal to 1, and $q = 2q_h + q_s + q_v$:

$$\begin{aligned}
\tilde{\mathbf{E}}^{A_1} &= q\mathbf{R}_0^{A_1} - q_h \mathbf{R}_h^{A_1} - q_h \mathbf{R}_{n-h}^{A_1} - q_s \mathbf{R}_n^{A_1} - q_v \mathbf{R}_{n+v}^{A_1} \\
&= q - 2q_h - q_s - q_v = 0.
\end{aligned} \tag{15}$$

The other three zero eigenvalues will come from $\tilde{\mathbf{E}}^{A_2}$ and the two copies of $\tilde{\mathbf{E}}^{E_1}$, because A_2 and E_1 are representations of the transformation of z- and xy-coordinates, as noted in Table 1. Substituting from Eq. (14) and Table 3 gives

$$\tilde{\mathbf{E}}^{A_2} = q - q_h - q_h - q_s(-1) - q_v(-1) = 2(q_s + q_v), \tag{16}$$

and

$$\begin{aligned}
\tilde{\mathbf{E}}^{E_1} &= q \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - q_h \begin{bmatrix} C_h & -S_h \\ S_h & C_h \end{bmatrix} - q_h \begin{bmatrix} C_{n-h} & -S_{n-h} \\ S_{n-h} & C_{n-h} \end{bmatrix} \\
&\quad - q_s \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - q_v \begin{bmatrix} C_v & S_v \\ S_v & -C_v \end{bmatrix} \\
&= 2q_h \begin{bmatrix} 1 - C_h & 0 \\ 0 & 1 - C_h \end{bmatrix} + q_s \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} + q_v \begin{bmatrix} 1 - C_v & -S_v \\ -S_v & 1 + C_v \end{bmatrix}. \tag{17}
\end{aligned}$$

To ensure that $\tilde{\mathbf{E}}^{A_2}$ and $\tilde{\mathbf{E}}^{E_1}$ are rank deficient, we require

$$\det(\tilde{\mathbf{E}}^{A_2}) = 0 \text{ and } \det(\tilde{\mathbf{E}}^{E_1}) = 0, \tag{18}$$

where $\det(\cdot)$ denotes determinant of a matrix. These conditions are met if the relations

between the force densities of different types of members are given by

$$q_v = -q_s, \quad \frac{q_h}{q_v} = +\frac{\sqrt{2 - 2C_v}}{2(1 - C_h)}. \tag{19}$$

Note that q_v and q_h are positive as cables carry tension (positive self-stress). The solution obtained here agrees with that by Connelly and Terrell (1995).

Using the force densities in Eq. (19), Eq. (14) can be rewritten as

$$\frac{1}{q_v}\tilde{\mathbf{E}}^\mu = 2t\mathbf{R}_0^\mu - t\mathbf{R}_h^\mu - t\mathbf{R}_{n-h}^\mu + \mathbf{R}_n^\mu - \mathbf{R}_{n+v}^\mu, \text{ with } t = q_h/q_v. \tag{20}$$

Eq. (19) can necessarily ensure a three-dimensional self-equilibrium configuration for a prismatic tensegrity structure, since $\tilde{\mathbf{E}}^{A_2}$ and $\tilde{\mathbf{E}}^{E_1}$ are singular. However, there is no guarantee that the structure is stable. To ensure the stability of the structure, we also need to investigate the positive definiteness of the other blocks.

From Eq. (20), the blocks $\tilde{\mathbf{E}}^{B_1}$ and $\tilde{\mathbf{E}}^{B_2}$ of representations B_1 and B_2 (when they exist for n even) are

$$\frac{1}{q_v}\tilde{\mathbf{E}}^{B_1} = (2 - (-1)^h - (-1)^{n-h})t + 1 - (-1)^v$$
$$\frac{1}{q_v}\tilde{\mathbf{E}}^{B_2} = (2 - (-1)^h - (-1)^{n-h})t - 1 + (-1)^v, \tag{21}$$

and the two-dimensional blocks $\tilde{\mathbf{E}}^{E_k}$ $(k = 1, \ldots, p)$ are

$$\frac{1}{q_v}\tilde{\mathbf{E}}^{E_k} = \begin{bmatrix} 2t(1 - C_{hk}) + 1 - C_{vk} & -S_{vk} \\ -S_{vk} & 2t(1 - C_{hk}) - (1 - C_{vk}) \end{bmatrix}. \tag{22}$$

The two eigenvalues of $\tilde{\mathbf{E}}^{E_k}$ are

$$\frac{\lambda_1^{E_k}}{q_v} = 2t(1 - C_{hk}) + \sqrt{2(1 - C_{vk})}, \quad \frac{\lambda_2^{E_k}}{q_v} = 2t(1 - C_{hk}) - \sqrt{2(1 - C_{vk})}. \tag{23}$$

$\lambda_1^{E_k} > 0$ holds since $t > 0$, $1 - C_{hk} > 0$ and $1 - C_{vk} > 0$. For representation E_1, we know from Eq. (19) that $\lambda_2^{E_1} = 0$. To satisfy positive semi-definiteness and minimum rank deficiency of the force density matrix, which are the two sufficient conditions for super stability of tensegrity structures (Connelly, 1999; Zhang and Ohsaki, 2007), $\lambda_2^{E_k}$ for $k > 1$ should be positive. Connelly and Terrell (1995) obtained the same two-dimensional blocks making use of the special properties of the force density matrix as a circulant matrix, and further proved that all other two-dimensional blocks (for $k > 1$) are positive definite if and only if $h = 1$; i.e., horizontal cables are connected to adjacent nodes.

To confirm that $h = 1$ is the super stability condition for prismatic tensegrity structures, we need to investigate the one-dimensional blocks: $\tilde{\mathbf{E}}^{A_1} = \tilde{\mathbf{E}}^{A_2} = 0$ always holds as discussed previously; and $\tilde{\mathbf{E}}^{B_1}$ and $\tilde{\mathbf{E}}^{B_2}$ exist only when n is even, for which we have the following relation from Eq. (21) for $h = 1$

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$$\frac{1}{q_v}\tilde{\mathbf{E}}^{B_1} = (2 - (-1)^h - (-1)^{n-h})t + 1 - (-1)^v = 4t + 1 - (-1)^v \geq 4t > 0,$$

$$\frac{1}{q_v}\tilde{\mathbf{E}}^{B_2} = (2 - (-1)^h - (-1)^{n-h})t - 1 + (-1)^v = 4t - 1 - (-1)^v$$

$$= 2\frac{\sqrt{2 - 2C_v}}{1 - C_1} - 1 - (-1)^v \geq 2\frac{\sqrt{2 - 2C_1}}{1 - C_1} - 1 - (-1)^1 = \frac{2\sqrt{2}}{\sqrt{1 - C_1}} - 2 > 0.$$

In summary, $h = 1$ guarantees the two sufficient conditions for super stability of a prismatic tensegrity structure: its force density matrix has rank deficiency of four (one in $\tilde{\mathbf{E}}^{A_1}$, one in $\tilde{\mathbf{E}}^{A_2}$ and two in the two copies of $\tilde{\mathbf{E}}^{E_1}$), which is the minimum value for a non-degenerate structure in three-dimensional space; and the force density matrix is positive semi-definite.

4. Symmetry-adapted Geometrical Stiffness Matrix

In this section, we present a direct strategy to find the symmetry-adapted geometrical stiffness matrix, which is used in Part I to investigate prestress stability of the structures.

As any node of a symmetric prismatic tensegrity structure is transformed to a different node by any symmetry operation of $\mathbf{D}_n$ except for the identity operation, by which all (in total $2n$) nodes remain unchanged, the traces of permutation matrices for nodes are summarised as follows (Fowler and Guest, 2000)

$$\Gamma(N) = \{2n, 0, \ldots, 0; \ 0, \ldots, 0\}$$
$$= A_1 + A_2 + (B_1 + B_2) + 2\sum_{k=1}^{p} E_k, \tag{24}$$

which has the same linear combination of irreducible representations as $\Gamma(E)$. Hence, the permutation matrices for nodes can be block-diagonalised in the same form as $\mathbf{R}_i$.

A representation of nodal coordinates of the tensegrity $\Gamma(D)$ in the external coordinate system can be found from the permutation representation of the nodes $\Gamma(N)$, multiplied by the representation of displacements of a single node, $\Gamma(T) = E_1 + A_2$ (noted in Table 1)

$$\Gamma(D) = \Gamma(N) \times \Gamma(T)$$
$$= \left(A_1 + A_2 + (B_1 + B_2) + 2\sum_{k=1}^{p} E_k\right) \times (E_1 + A_2)$$
$$= 3A_1 + 3A_2 + (3B_1 + 3B_2) + 6\sum_{k=1}^{p} E_k, \tag{25}$$

where $\times$ denotes direct product (table of direct product of two representations of dihedral group can be found in many textbooks on group representation theory, e.g., the book by Atkins $et\ al.$ (1970)). Similar to $\Gamma(E)$, $\Gamma(D)$ in Eq. (25) characterises the structure of the symmetry-adapted geometrical stiffness matrix $\tilde{\mathbf{S}}$: the blocks corresponding to the one- and two-dimensional representations are 3-by-3 and 6-by-6 matrices, respectively.

Table 4
Reducible representation matrices for external coordinate system.

$$\mathbf{N}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{N}_h = \begin{bmatrix} C_h & -S_h & 0 \\ S_h & C_h & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{N}_{n-h} = \begin{bmatrix} C_h & S_h & 0 \\ -S_h & C_h & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{N}_n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad \mathbf{N}_{n+v} = \begin{bmatrix} C_v & S_v & 0 \\ S_v & -C_v & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

From Eq. (25), the reducible representation matrix $\bar{\mathbf{R}}_i^\mu$ of the i-th symmetry operation for nodal coordinates can be formulated as follows, using the direct sum of irreducible matrices of E_1 and A_2:

$$\bar{\mathbf{R}}_i^\mu = \mathbf{R}_i^\mu \otimes \begin{bmatrix} \mathbf{R}_i^{E_1} & 0 \\ \mathbf{O} & \mathbf{R}_i^{A_2} \end{bmatrix} = \mathbf{R}_i^\mu \otimes \mathbf{N}_i, \tag{26}$$

where $\mathbf{N}_i$ for $i \in \{0, h, n-h, n, n+v\}$ are listed in Table 4.

Similar to the symmetry-adapted force density matrix, we have the following lemma for the symmetry-adapted geometrical stiffness matrix $\tilde{\mathbf{S}}$.

Lemma 2 *The blocks $\tilde{\mathbf{S}}^\mu$ of the symmetry-adapted geometrical stiffness matrix $\tilde{\mathbf{S}}$, corresponding to the representation μ, can be written in a general form as*

$$\frac{1}{q_v}\tilde{\mathbf{S}}^\mu = 2t\bar{\mathbf{R}}_0^\mu - t\bar{\mathbf{R}}_h^\mu - t\bar{\mathbf{R}}_{n-h}^\mu + \bar{\mathbf{R}}_n^\mu - \bar{\mathbf{R}}_{n+v}^\mu. \tag{27}$$

Proof. As shown in Eq. (25), the representation of nodal coordinates $\Gamma(D)$ is the multiplication of that of permutation representation of nodes $\Gamma(N)$ and that of displacements of a single node $\Gamma(T)$; moreover, the permutation matrices of nodes can be block-digonalised in the same form as $\mathbf{R}_i^\mu$. Hence, the symmetry-adapted geometrical stiffness matrix corresponding to the irreducible representation μ can be found as follows

$$\frac{1}{q_v}\tilde{\mathbf{S}}^\mu = 2t\bar{\mathbf{R}}_0^\mu - t\bar{\mathbf{R}}_h^\mu - t\bar{\mathbf{R}}_{n-h}^\mu + \bar{\mathbf{R}}_n^\mu - \bar{\mathbf{R}}_{n+v}^\mu,$$

where $\bar{\mathbf{R}}_i^\mu$ is Kronecher product of the irreducible representation matrix of the permutation of nodes $\mathbf{R}_i^\mu$ and the reducible representation matrix of the coordinates of a single node $\mathbf{N}_i$ as defined in Eq. (26).

Thus, the lemma is proved. □

From Eq. (27), the blocks $\tilde{\mathbf{S}}^{A_1}$ and $\tilde{\mathbf{S}}^{A_2}$ can be written as

Table 5
Relationships between eigenvalues of symmetry-adapted forms of the geometrical stiffness matrix and the force density matrix.

$\tilde{\mathbf{S}}^\mu$	$\tilde{\mathbf{E}}^\mu$						n
$3A_1$		A_2		E_1			
$3A_2$	A_1			E_1			
$3B_1$			B_2			E_p	even
$3B_2$		B_1				E_p	even
$6E_1$	A_1	A_2		E_1	E_2		
$6E_k$				E_{k-1}	E_k	E_{k+1}	
$6E_p$					E_{p-1}	$2E_p$	odd
$6E_p$			B_1	B_2	E_{p-1}	E_p	even

$$\frac{1}{q_v}\tilde{\mathbf{S}}^{A_1} = \begin{bmatrix} 2t(1-C_h)+1-C_v & -S_v & 0 \\ -S_v & 2t(1-C_h)-1+C_v & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{1}{q_v}\tilde{\mathbf{S}}^{A_2} = \begin{bmatrix} 2t(1-C_h)-1+C_v & S_v & 0 \\ S_v & 2t(1-C_h)+1-C_v & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{28}$$

It is easy to verify that the eigenvalues of $\tilde{\mathbf{S}}^{A_1}$ are those of $\left(\tilde{\mathbf{E}}^{E_1} \oplus \tilde{\mathbf{E}}^{A_2}\right)$; similarly, the eigenvalues of $\tilde{\mathbf{S}}^{A_2}$ are those of $\left(\tilde{\mathbf{E}}^{E_1} \oplus \tilde{\mathbf{E}}^{A_1}\right)$, which follows from

$$\begin{aligned} A_1 \times (E_1 + A_2) &= E_1 + A_2 \\ A_2 \times (E_1 + A_2) &= E_1 + A_1. \end{aligned} \tag{29}$$

In a similar way, relationships between the eigenvalues of symmetry-adapted forms of the force density matrix and the geometrical stiffness matrices are summarised in Table 5.

Blocks $\tilde{\mathbf{S}}^{B_1}$ and $\tilde{\mathbf{S}}^{B_2}$, when they exist for n even, are given by

$$\frac{1}{q_v}\tilde{\mathbf{S}}^{B_1} = \begin{bmatrix} \phi_1 & (-1)^{v+1}S_v & 0 \\ (-1)^{v+1} & \phi_2 & 0 \\ 0 & 0 & \phi_3 \end{bmatrix}$$

$$\frac{1}{q_v}\tilde{\mathbf{S}}^{B_2} = \begin{bmatrix} \phi_4 & (-1)^{v}S_v & 0 \\ (-1)^{v} & \phi_5 & 0 \\ 0 & 0 & \phi_6 \end{bmatrix}, \tag{30}$$

where

$$\begin{aligned} \phi_1 &= 2t(1-(-1)^hC_h)+1-(-1)^vC_v & \phi_2 &= 2t(1-(-1)^hC_h)-1+(-1)^vC_v \\ \phi_3 &= 2t(1-(-1)^h)-1+(-1)^v & \phi_4 &= 2t(1-(-1)^hC_h)+1+(-1)^vC_v \\ \phi_5 &= 2t(1-(-1)^hC_h)-1-(-1)^vC_v & \phi_6 &= 2t(1-(-1)^h)-1-(-1)^v. \end{aligned} \tag{31}$$

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And blocks $\tilde{\mathbf{S}}^{E_k}$ of the two-dimensional representations E_k are given by

$$\frac{1}{q_v}\tilde{\mathbf{S}}^{E_k} = \begin{bmatrix} \varphi_1 + \varphi_2 & -\varphi_3 & 0 & -\varphi_4 & -\varphi_5 - \varphi_6 & 0 \\ -\varphi_3 & \varphi_1 - \varphi_2 & 0 & \varphi_5 - \varphi_6 & \varphi_4 & 0 \\ 0 & 0 & \varphi_7 - \varphi_8 & 0 & 0 & \varphi_9 \\ -\varphi_4 & \varphi_5 - \varphi_6 & 0 & \varphi_1 - \varphi_2 & \varphi_3 & 0 \\ -\varphi_5 - \varphi_6 & \varphi_4 & 0 & \varphi_3 & \varphi_1 + \varphi_2 & 0 \\ 0 & 0 & \varphi_9 & 0 & 0 & \varphi_7 + \varphi_8 \end{bmatrix}, \tag{32}$$

where

$$\begin{array}{lll} \varphi_1 = 2t(1 - C_h C_{hk}) & \varphi_2 = 1 - C_v C_{vk} & \varphi_3 = S_v C_{vk} \\ \varphi_4 = C_v S_{vk} & \varphi_5 = 2t S_h S_{hk} & \varphi_6 = S_v S_{vk} \\ \varphi_7 = 2t(1 - C_{hk}) & \varphi_8 = 1 - C_{vk} & \varphi_9 = S_{vk}. \end{array} \tag{33}$$

5. Symmetry-adapted Equilibrium Matrix

This section presents the symmetry-adapted equilibrium matrix $\tilde{\mathbf{A}}$ and the mechanisms $\tilde{\mathbf{M}}$ lying in the null-space of its transpose $\tilde{\mathbf{A}}^{\mathrm{T}}$.

5.1 Block Structure

Unlike the force density matrix $\mathbf{E}$ or the geometrical stiffness matrix $\mathbf{S}$, the equilibrium matrix $\mathbf{A} \in \Re^{6n \times 4n}$ of a prismatic tensegrity structure is not square. In conventional methods, the symmetry-adapted equilibrium matrix $\tilde{\mathbf{A}}$ can be computed as follows using the transformation matrices $\mathbf{T}_D$ and $\mathbf{T}_M$ respectively for external and internal coordinate systems (see, for example, Kangwai and Guest (2000))

$$\underset{6n \times 4n}{\tilde{\mathbf{A}}} = \underset{6n \times 6n}{\mathbf{T}_D} \mathbf{A} (\underset{4n \times 4n}{\mathbf{T}_M})^{\mathrm{T}}. \tag{34}$$

To make clear the structure of $\tilde{\mathbf{A}}$, we initially investigate linear combination of representations of its members (internal coordinates). Using tables of characters similar to those in Kangwai and Guest (2000), it should be noted that different types of members cannot be transformed to each other by any symmetry operation. Thus, the horizontal cables, struts and vertical cables should be considered separately.

Tables 6, 7 and 8 show the numbers of members unshifted by each symmetry operation. Each linear combination of representations is also written in terms of irreducible representations. $\Gamma(M)$ is the linear combination of representations for all members, $\Gamma(M) = \Gamma(M_h) + \Gamma(M_s) + \Gamma(M_v)$ where $\Gamma(M_h)$, $\Gamma(M_s)$ and $\Gamma(M_v)$ are those for horizontal cables, struts and vertical cables, respectively. The tables are formulated separately for n odd (see for example Fig. 2); n even and v odd (see for example Fig. 3.(a)); n even and v even (see for example Fig. 3.(b)).

Based on $\Gamma(M)$ and $\Gamma(D)$, we can describe the size of the blocks in the symmetry-adapted equilibrium matrix $\tilde{\mathbf{A}}$. There are a blocks for each a-dimensional irreducible representation. The number of rows of a (block) matrix is given by the coefficient on the corresponding representation in $\Gamma(D)$, and the number of columns by the coefficient in $\Gamma(M)$. Thus, for example, the A_1 block $\tilde{\mathbf{A}}^{A_1}$ is a 3-by-3 matrix, because there are three A_1 representations in both of $\Gamma(D)$ and $\Gamma(M)$. Furthermore, columns of $\tilde{\mathbf{A}}^{A_1}$ come from
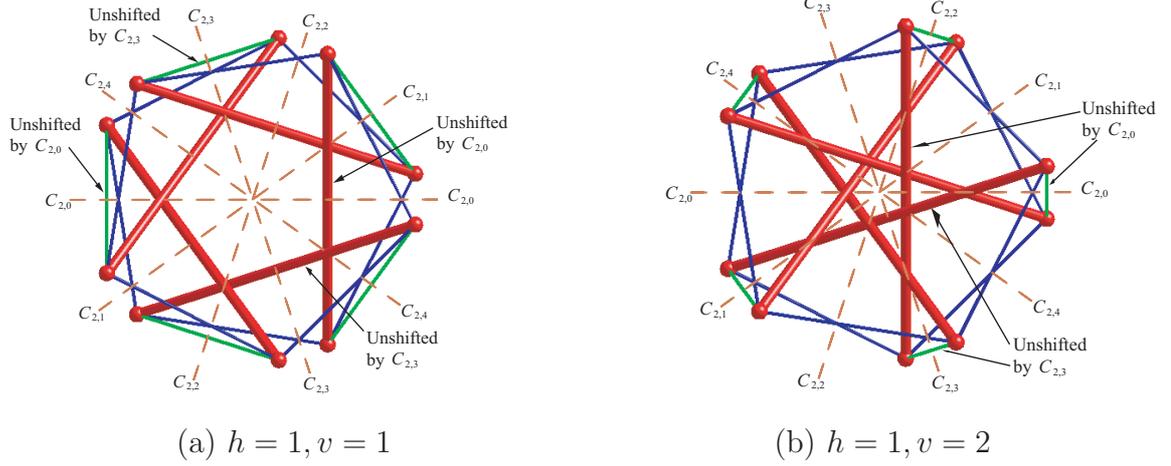
(a) $h = 1, v = 1$

(b) $h = 1, v = 2$

Fig. 2. Structures $\mathbf{D}_n^{h,v}$ with $n(=5)$ odd. One strut and one vertical cable remain unshifted by any two-fold rotations, and all are shifted by any n-fold rotations except the identity operation.
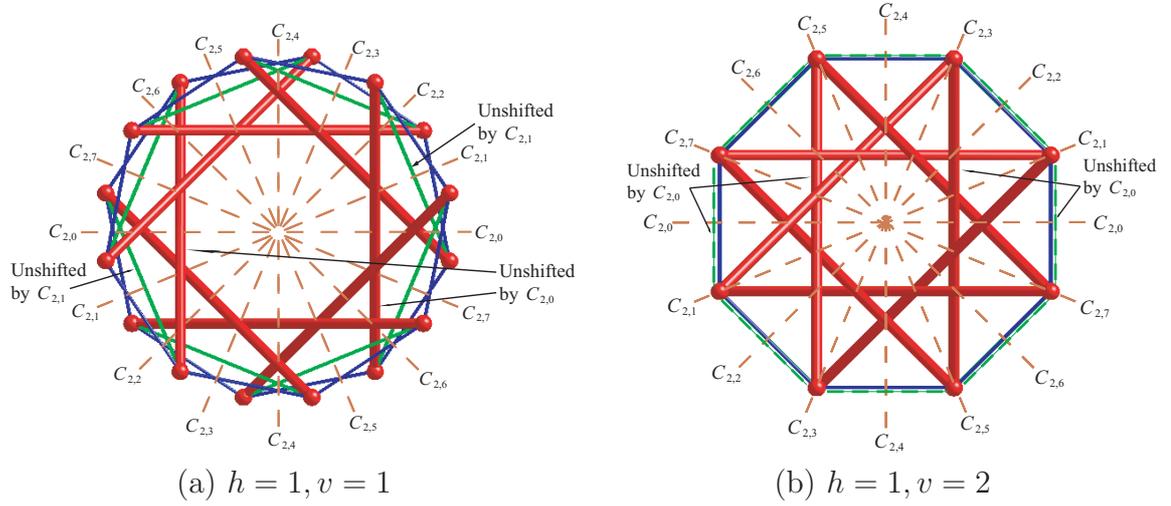

(a) $h = 1, v = 1$

(b) $h = 1, v = 2$

Fig. 3. Structures $\mathbf{D}_n^{h,v}$ with $n(=8)$ even. Two struts remain unshifted by a two-fold rotation $C_{2,2i}$, and all struts are shifted by $C_{2,2i+1}$. For v odd, all vertical cables are shifted by $C_{2,2i}$, and two are unchanged by $C_{2,2i+1}$. For v even, two vertical cables are unshifted by $C_{2,2i}$, and all are shifted by $C_{2,2i+1}$.

the horizontal cables, struts and vertical cables separately, because all of $\Gamma(M_h)$, $\Gamma(M_s)$ and $\Gamma(M_v)$ have one representation A_1. Similarly, $\tilde{\mathbf{A}}^{A_2}$ is a 3-by-1 matrix, and the only column comes from the horizontal cables because there is no representation A_2 for struts or vertical cables.

5.2 Representation of Mechanisms

Following Fowler and Guest (2000), the shape of blocks of the equilibrium matrix gives information about mechanisms and states of self-stress. Written in terms of the representation of mechanisms including rigid-body motions $\Gamma(m)$ and the self-stress $\Gamma(s)$, we have $\Gamma(m) - \Gamma(s) = \Gamma(D) - \Gamma(M)$. However, we have shown that the prismatic tensegrity structures have only one mode of symmetric self-stress, thus $\Gamma(s) = A_1$. Therefore, the

Table 6
The linear combination of representations of members for n odd.

	E	C_n^i	$C_{2,i}$	representation
$\Gamma(M_h)$	$2n$	0	0	$A_1 + A_2 + 2\sum_{k=1}^{p} E_k$
$+\Gamma(M_s)$	n	0	1	$A_1 + \sum_{k=1}^{p} E_k$
$+\Gamma(M_v)$	n	0	1	$A_1 + \sum_{k=1}^{p} E_k$
$= \Gamma(M)$				$3A_1 + A_2 + 4\sum_{k=1}^{p} E_k$

Table 7
The linear combination of representations of members for n even and v odd.

	E	C_n^i	$C_{2,2i}$	$C_{2,2i+1}$	representation
$\Gamma(M_h)$	$2n$	0	0	0	$A_1 + A_2 + B_1 + B_2 + 2\sum_{k=1}^{p} E_k$
$+\Gamma(M_s)$	n	0	2	0	$A_1 + B_1 + \sum_{k=1}^{p} E_k$
$+\Gamma(M_v)$	n	0	0	2	$A_1 + B_2 + \sum_{k=1}^{p} E_k$
$= \Gamma(M)$					$3A_1 + A_2 + 2B_1 + 2B_2 + 4\sum_{k=1}^{p} E_k$

Table 8
The linear combination of representation of members for n even and v even.

	E	C_n^i	$C_{2,2i}$	$C_{2,2i+1}$	representation
$\Gamma(M_h)$	$2n$	0	0	0	$A_1 + A_2 + B_1 + B_2 + 2\sum_{k=1}^{p} E_k$
$+\Gamma(M_s)$	n	0	2	0	$A_1 + B_1 + \sum_{k=1}^{p} E_k$
$+\Gamma(M_v)$	n	0	2	0	$A_1 + B_1 + \sum_{k=1}^{p} E_k$
$= \Gamma(M)$					$3A_1 + A_2 + 3B_1 + B_2 + 4\sum_{k=1}^{p} E_k$

representation of the mechanisms is

$$\Gamma(m) = A_1 + \Gamma(D) - \Gamma(M); \tag{35}$$

for v even

$$\Gamma(m) = A_1 + (3A_1 + 3A_2 + (3B_1 + 3B_2+)6\sum_{k=1}^{p} E_k) - (3A_1 + A2 + (2B_1 + 2B_2+)4\sum_{k=1}^{p} E_k)$$

$$= A_1 + 2A_2 + (2B_2) + 2\sum_{k=1}^{p} E_k,$$

and for v odd

$$\Gamma(m) = A_1 + (3A_1 + 3A_2 + (3B_1 + 3B_2+)6\sum_{k=1}^{p} E_k) - (3A_1 + A2 + (3B_1 + B_2+)4\sum_{k=1}^{p} E_k)$$

$$= A_1 + 2A_2 + (B_1 + B_2) + 2\sum_{k=1}^{p} E_k,$$

Hence, there is one mechanism of A_1 symmetry, two mechanisms of A_2 symmetry and so on. In total, there are $2n + 1$ mechanisms including the rigid-body motions.

5.3 Unitary Member Direction

The concept of unitary member direction introduced in this subsection has a vital role in deriving the symmetry-adapted equilibrium matrix, and thus, infinitesimal mechanisms.
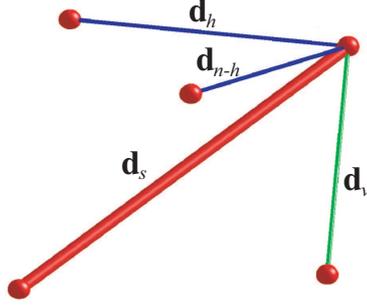
Fig. 4. A single node of prismatic tensegrity structures with dihedral symmetry. The node is connected by two horizontal cables, one vertical cable and one strut.

The equilibrium matrix $\mathbf{A}$ can be formulated as follows (Zhang and Ohsaki, 2006)

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}^x \\ \mathbf{A}^y \\ \mathbf{A}^z \end{bmatrix} = \begin{bmatrix} \mathbf{C}^{\mathrm{T}}\mathbf{U}\mathbf{L}^{-1} \\ \mathbf{C}^{\mathrm{T}}\mathbf{V}\mathbf{L}^{-1} \\ \mathbf{C}^{\mathrm{T}}\mathbf{W}\mathbf{L}^{-1} \end{bmatrix}, \tag{36}$$

where $\mathbf{C} \in \Re^{4n \times 2n}$ describes the connectivity of the structure; $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}$ ($\in \Re^{4n \times 4n}$) are diagonal matrices, in which the diagonal entries are coordinate differences in each of directions x, y and z; and $\mathbf{L} \in \Re^{4n \times 4n}$ is a diagonal matrix, of which diagonal entries are member lengths. Hence, the diagonal entries of $\mathbf{U}\mathbf{L}^{-1}$, $\mathbf{V}\mathbf{L}^{-1}$, and $\mathbf{W}\mathbf{L}^{-1}$ are the x, y and z components of the unitary member directions.

When we apply transformation matrices to $\mathbf{A}$ to derive its symmetry-adapted form $\tilde{\mathbf{A}}$, as in Eq. (34), we are thus actually dealing with the unitary member directions, and in fact the symmetry-adapted equilibrium matrix can be directly derived from these unitary member directions.

Consider a single reference node of the structure as shown in Fig. 4. The coordinate of the reference node can be written as follows (Zhang *et al.*, 2008a)

$$\mathbf{X}_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} C_v - 1 + \sqrt{2(1 - C_v)} \\ S_v \\ \frac{H}{2} \end{bmatrix} = \begin{bmatrix} 2S_{v/2}(1 - S_{v/2}) \\ S_v \\ \frac{H}{2} \end{bmatrix}, \tag{37}$$

where H denotes height(-to-radius ratio) of the structure. Other nodes of the structure can be determined using symmetry operations.

Denote the lengths of the strut, horizontal cable and vertical cable as l_s, l_h and l_v, respectively. The unitary directions $\mathbf{d}_h$ and $\mathbf{d}_{n-h}$ of the two horizontal cables connected to the reference node can be computed as

$$l_h \mathbf{d}_h = \mathbf{X}_0 - \mathbf{N}_h \mathbf{X}_0 = \begin{bmatrix} x_0 - C_h x_0 + S_h y_0 \\ y_0 - C_h y_0 - S_h x_0 \\ 0 \end{bmatrix} = 4 S_{\frac{h}{2}} S_{\frac{v}{2}} \begin{bmatrix} C_{\frac{h+v}{2}} + S_{\frac{h}{2}} \\ S_{\frac{h+v}{2}} - C_{\frac{h}{2}} \\ 0 \end{bmatrix},$$

$$l_h \mathbf{d}_{n-h} = \mathbf{X}_0 - \mathbf{N}_{n-h} \mathbf{X}_0 = \begin{bmatrix} x_0 - C_h x_0 - S_h y_0 \\ y_0 - C_h y_0 + S_h x_0 \\ 0 \end{bmatrix} = 4 S_{\frac{h}{2}} S_{\frac{v}{2}} \begin{bmatrix} -C_{\frac{h-v}{2}} + S_{\frac{h}{2}} \\ S_{\frac{h-v}{2}} + C_{\frac{h}{2}} \\ 0 \end{bmatrix}. \quad (38)$$

$$(39)$$

Thus,

$$\mathbf{d}_h + \mathbf{d}_{n-h} = \frac{8 S_{\frac{h}{2}}^2 S_{\frac{v}{2}}}{l_h} \begin{bmatrix} 1 - S_{\frac{v}{2}} \\ C_{\frac{v}{2}} \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{d}_h - \mathbf{d}_{n-h} = \frac{4 S_h S_{\frac{v}{2}}}{l_h} \begin{bmatrix} C_{\frac{v}{2}} \\ S_{\frac{v}{2}} - 1 \\ 0 \end{bmatrix}. \quad (40)$$

Unitary directions of the strut $\mathbf{d}_s$ and vertical cable $\mathbf{d}_v$ are

$$l_s \mathbf{d}_s = \mathbf{X}_0 - \mathbf{N}_n \mathbf{X}_0 = \begin{bmatrix} 0 \\ 2y_0 \\ H \end{bmatrix} = \begin{bmatrix} 0 \\ 2S_v \\ H \end{bmatrix},$$

$$l_v \mathbf{d}_v = \mathbf{X}_0 - \mathbf{N}_{n+v} \mathbf{X}_0 = \begin{bmatrix} x_0 - C_v x_0 - S_v y_0 \\ y_0 + C_v y_0 - S_v x_0 \\ H \end{bmatrix} = 4 S_{\frac{v}{2}} (1 - S_{\frac{v}{2}}) \begin{bmatrix} -S_{\frac{v}{2}} \\ C_{\frac{v}{2}} \\ \bar{H} \end{bmatrix}, \quad (41)$$

where $\bar{H} = H/[4 S_{\frac{v}{2}} (1 - S_{\frac{v}{2}})]$.

5.4 Symmetry-adapted Equilibrium Matrix

Because horizontal cables have one-to-one correspondence with the symmetry operations of a dihedral group, its transformation matrix $_h \mathbf{T}_M^\mu$ for a one-dimensional representation μ can be defined as (Kettle, 1995)

$$_h \mathbf{T}_M^\mu = \frac{1}{\sqrt{2n}} \left[\mathbf{R}_0^\mu, \ldots, \mathbf{R}_j^\mu, \ldots, \mathbf{R}_{2n-1}^\mu \right]. \quad (42)$$

Since the i-th and $(n+i)$-th ($i = \in \{0, \ldots, n-1\}$) symmetry operations take the strut connected by nodes N_i and N_{n+i} to itself, the transformation matrix $_s \mathbf{T}_M^\mu$ for struts can be written as

$$_s \mathbf{T}_M^\mu = \frac{1}{\sqrt{4n}} [\mathbf{R}_0^\mu + \mathbf{R}_n^\mu, \ldots, \mathbf{R}_i^\mu + \mathbf{R}_{n+i}^\mu, \ldots, \mathbf{R}_{n-1}^\mu + \mathbf{R}_{2n-1}^\mu]; \quad (43)$$

and similarly, $_v \mathbf{T}_M^\mu$ for vertical cables is

$$_v \mathbf{T}_M^\mu = \frac{1}{\sqrt{4n}} [\mathbf{R}_0^\mu + \mathbf{R}_{n+v}^\mu, \ldots, \mathbf{R}_i^\mu + \mathbf{R}_{n+i+v}^\mu, \ldots, \mathbf{R}_{n-1-v}^\mu + \mathbf{R}_{2n-1}^\mu, \mathbf{R}_{n-v}^\mu + \mathbf{R}_n^\mu, \ldots, \mathbf{R}_{n-1}^\mu + \mathbf{R}_{n-1+v}^\mu].$$

$$(44)$$

Note that there are $2n$ entries in $_h \mathbf{T}_M^\mu$ but only n entries in $_s \mathbf{T}_M^\mu$ and $_v \mathbf{T}_M^\mu$, which are identical to the number of the members.

For a two-dimensional representation E_k $(k = 1, \ldots, p)$, the transformation matrix $_h\mathbf{T}_M^{E_k} \in \Re^{4 \times 2n}$ for horizontal cables is given as

$$
_h\mathbf{T}_M^{E_k} = \frac{1}{\sqrt{n}} \begin{bmatrix} R_0^\mu(1,1), & \ldots, & R_i^\mu(1,1), & \ldots, & R_{2n-1}^\mu(1,1) \\ R_0^\mu(1,2), & \ldots, & R_i^\mu(1,2), & \ldots, & R_{2n-1}^\mu(1,2) \\ R_0^\mu(2,1), & \ldots, & R_i^\mu(2,1), & \ldots, & R_{2n-1}^\mu(2,1) \\ R_0^\mu(2,2), & \ldots, & R_i^\mu(2,2), & \ldots, & R_{2n-1}^\mu(2,2) \end{bmatrix} = \frac{1}{\sqrt{n}} \begin{bmatrix} [C_{ik}], & [C_{ik}] \\ -[S_{ik}], & [S_{ik}] \\ [S_{ik}], & [S_{ik}] \\ [C_{ik}], & -[C_{ik}] \end{bmatrix},
$$
(45)

where $[C_{ik}]$ and $[S_{ik}]$ $(\in \Re^n)$ are row vectors:

$$
[C_{ik}] = \begin{bmatrix} C_0, C_k, \ldots, C_{ik}, \ldots, C_{(n-1)k} \end{bmatrix}
$$
$$
[S_{ik}] = \begin{bmatrix} S_0, S_k, \ldots, S_{ik}, \ldots, S_{(n-1)k} \end{bmatrix}, \qquad \text{for } i = 0, \ldots, n-1.
$$

For the struts and vertical cables, the transformation matrices are written as follows

$$
_s\bar{\mathbf{T}}_M^{E_k} = \begin{bmatrix} [C_{jk} + C_{jk}] \\ -[S_{jk} + S_{jk}] \\ [S_{jk} + S_{jk}] \\ [C_{jk} + (-C_{jk})] \end{bmatrix} = \begin{bmatrix} [2C_{jk}] \\ [0] \\ [2S_{jk}] \\ [0] \end{bmatrix},
$$

$$
_v\bar{\mathbf{T}}_M^{E_k} = \begin{bmatrix} \left[C_{jk} + C_{(j+v)k} \right] \\ -\left[S_{jk} + S_{(j+v)k} \right] \\ \left[S_{jk} + S_{(j+v)k} \right] \\ \left[C_{jk} + (-C_{(j+v)k}) \right] \end{bmatrix} = 2 \begin{bmatrix} \left[C_{(j+\frac{1}{2}v)k} C_{\frac{vk}{2}} \right] \\ -\left[C_{(j+\frac{1}{2}v)k} S_{\frac{vk}{2}} \right] \\ \left[S_{(j+\frac{1}{2}v)k} C_{\frac{vk}{2}} \right] \\ \left[S_{(j+\frac{1}{2}v)k} S_{\frac{vk}{2}} \right] \end{bmatrix} \quad \text{for } j = 0, \ldots, n-1.
$$
(46)

It is apparent that the first and the second rows, and the third and the fourth rows are dependent, hence, only the first and the third rows are present in $_s\mathbf{T}_M^{E_k}$ and $_h\mathbf{T}_M^\mu$ $(\in \Re^{2 \times n})$, which are normalised as

$$
_s\mathbf{T}_M^{E_k} = \frac{1}{\sqrt{2n}} \begin{bmatrix} [2C_{jk}] \\ [2S_{jk}] \end{bmatrix},
$$

$$
_v\mathbf{T}_M^{E_k} = \frac{1}{\sqrt{n(1 + C_{vk})}} \begin{bmatrix} \left[C_{jk} + C_{(j+v)k} \right] \\ \left[S_{jk} + S_{(j+v)k} \right] \end{bmatrix} \quad \text{for } j = 0, \ldots, n-1.
$$
(47)

The transformation matrix $\mathbf{T}_M^\mu$ of the members for the representation μ can then be combined as

$$
\mathbf{T}_M^\mu = \begin{bmatrix} _h\mathbf{T}_M^\mu & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & _s\mathbf{T}_M^\mu & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & _v\mathbf{T}_M^\mu \end{bmatrix},
$$
(48)

and the transformation matrix $\mathbf{T}_M \in \Re^{4n \times 4n}$ of the members for all representations can be further assembled as

$$
\mathbf{T}_M = \begin{bmatrix} \mathbf{T}_M^{A_1} \\ \mathbf{T}_M^{A_2} \\ \vdots \\ \mathbf{T}_M^{E_p} \end{bmatrix}.
$$
(49)

As indicated in the formulations of the equilibrium matrix $\mathbf{A}$ and its transformation matrices, components of its symmetry-adapted form $\tilde{\mathbf{A}}$ can be separately formulated for different types of members. Since horizontal cables have one-to-one correspondence with symmetry operations, their symmetry-adapted components $\tilde{\mathbf{A}}_h^\mu$ for representation μ can be directly formulated as follows using its unitary member directions $\mathbf{d}_h$ and $\mathbf{d}_{n-h}$

$$\tilde{\mathbf{A}}_h^\mu = \mathbf{R}_0^\mu \otimes \mathbf{d}_h + \mathbf{R}_h^\mu \otimes \mathbf{d}_{n-h}, \tag{50}$$

in a similar way to the formulation of the symmetry-adapted geometrical stiffness matrix presented in Section 4.

The symmetry-adapted components $\tilde{\mathbf{A}}_s^\mu$ of struts and $\tilde{\mathbf{A}}_v^\mu$ of vertical cables can be formulated in a manner similar to that of horizontal cables in Eq. (50), as follows

$$\tilde{\mathbf{A}}_s^\mu = \frac{1}{a_s^\mu}(\hat{\mathbf{R}}_0^\mu + \hat{\mathbf{R}}_n^\mu) \otimes \mathbf{d}_s \text{ and } \tilde{\mathbf{A}}_v^\mu = \frac{1}{a_v^\mu}(\hat{\mathbf{R}}_v^\mu + \hat{\mathbf{R}}_{n+v}^\mu) \otimes \mathbf{d}_v, \tag{51}$$

where $a_s^\mu = a_v^\mu = \sqrt{2}$ for one-dimensional representations and $a_s^\mu = \sqrt{2}$, $a_v^\mu = \sqrt{1 + C_{vk}}$ for two-dimensional representations, to ensure correct normalisation. $\hat{\mathbf{R}}_j^\mu$ is constructed from components of the irreducible representation matrix of $\mathbf{R}_j^\mu$: for one-dimensional representations $\hat{\mathbf{R}}_j^\mu = \mathbf{R}_j^\mu$, and for two-dimensional representations

$$\hat{\mathbf{R}}_j^{E_k} = \begin{bmatrix} C_{jk} \\ S_{jk} \end{bmatrix}. \tag{52}$$

For convenience, we write the symmetry-adapted components of all types of members corresponding to each representation μ together in a form as

$$\tilde{\mathbf{A}}^\mu = \begin{bmatrix} \tilde{\mathbf{A}}_h^\mu, & \tilde{\mathbf{A}}_s^\mu, & \tilde{\mathbf{A}}_v^\mu \end{bmatrix}. \tag{53}$$

Consider initially A_1. Since all types of members have representation A_1, $\tilde{\mathbf{A}}^{A_1}$ can be formulated as

$$\tilde{\mathbf{A}}_{3\times3}^{A_1} = \begin{bmatrix} \mathbf{d}_h + \mathbf{d}_{n-h}, & \frac{1+1}{\sqrt{2}}\mathbf{d}_s, & \frac{1+1}{\sqrt{2}}\mathbf{d}_v \end{bmatrix} = \begin{bmatrix} \mathbf{d}_h + \mathbf{d}_{n-h}, & \sqrt{2}\mathbf{d}_s, & \sqrt{2}\mathbf{d}_v \end{bmatrix}. \tag{54}$$

$\tilde{\mathbf{A}}^{A_1}$ is singular with rank deficiency of one, and the infinitesimal mechanism lying in its transpose is

$$\mathbf{m}^{A_1} = \begin{bmatrix} 4S_{\frac{v}{2}}^2 \\ -1 \\ 1 \end{bmatrix}. \tag{55}$$

Now consider the block of the equilibrium matrix corresponding to irreducible representation A_2. Only horizontal cables have an A_2 representation, thus

$$\tilde{\mathbf{A}}^{A_2} = \begin{bmatrix} \mathbf{d}_h + \mathbf{d}_{n-h} \end{bmatrix} = \frac{8S_{\frac{h}{2}}^2 S_{\frac{v}{2}}}{l_h} \begin{bmatrix} 1 - S_{\frac{v}{2}} \\ C_{\frac{v}{2}} \\ 0 \end{bmatrix}. \tag{56}$$

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The mechanisms in its null-space are indeed the rigid-body motions, translation in and rotation about the z-axis, as indicated in Table 1, and are found as follows

$$\mathbf{m}_1^{A_2} = \begin{bmatrix} 1 + S_{\frac{v}{2}} \\ -C_{\frac{v}{2}} \\ 0 \end{bmatrix} \text{ and } \mathbf{m}_2^{A_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{57}$$

When n is even, the struts have representation B_1; the horizontal cables have representation B_1 for v odd and B_2 for v even. Hence, for n even and v odd, we have

$$\tilde{\mathbf{A}}^{B_1} = \begin{bmatrix} \mathbf{d}_h + (-1)^h \mathbf{d}_{n-h}, & \sqrt{2}\mathbf{d}_s \end{bmatrix} \text{ and } \tilde{\mathbf{A}}^{B_2} = \begin{bmatrix} \mathbf{d}_h + (-1)^h \mathbf{d}_{n-h}, & \sqrt{2}\mathbf{d}_v \end{bmatrix}, \tag{58}$$

with infinitesimal mechanisms

$$\mathbf{m}^{B_1} = \begin{bmatrix} C_{\frac{v}{2}} \\ S_{\frac{v}{2}} + 1 \\ -\frac{2S_v(S_{\frac{v}{2}}+1)}{H} \end{bmatrix} \text{ and } \mathbf{m}^{B_2} = \begin{bmatrix} C_{\frac{v}{2}} \\ S_{\frac{v}{2}} + 1 \\ -\frac{S_v(1-S_{\frac{v}{2}})}{H} \end{bmatrix} \text{ for } h \text{ odd}, \tag{59}$$

and

$$\mathbf{m}^{B_1} = \begin{bmatrix} 1 + S_{\frac{v}{2}} \\ -C_{\frac{v}{2}} \\ \frac{2S_v C_{\frac{v}{2}}}{H} \end{bmatrix} \text{ and } \mathbf{m}^{B_2} = \begin{bmatrix} 1 + S_{\frac{v}{2}} \\ -C_{\frac{v}{2}} \\ \frac{2S_v C_{\frac{v}{2}}(2S_{\frac{v}{2}}-1)}{H} \end{bmatrix} \text{ for } h \text{ even}; \tag{60}$$

when both n and v are even, we have

$$\tilde{\mathbf{A}}^{B_1} = \begin{bmatrix} \mathbf{d}_h + (-1)^h \mathbf{d}_{n-h}, & \sqrt{2}\mathbf{d}_s, & \sqrt{2}\mathbf{d}_v \end{bmatrix} \text{ and } \tilde{\mathbf{A}}^{B_2} = \begin{bmatrix} \mathbf{d}_h + (-1)^h \mathbf{d}_{n-h} \end{bmatrix}, \tag{61}$$

with the infinitesimal mechanisms

$$\mathbf{m}_1^{B_2} = \begin{bmatrix} C_{\frac{v}{2}} \\ S_{\frac{v}{2}} + 1 \\ 0 \end{bmatrix} \text{ and } \mathbf{m}_2^{B_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ for } h \text{ odd}, \tag{62}$$

and

$$\mathbf{m}_1^{B_2} = \begin{bmatrix} 1 + S_{\frac{v}{2}} \\ C_{\frac{v}{2}} \\ 0 \end{bmatrix} \text{ and } \mathbf{m}_2^{B_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ for } h \text{ even}. \tag{63}$$

$\tilde{\mathbf{A}}^{E_k}$ for a two-dimensional representation E_k is

$$\tilde{\mathbf{A}}^{E_k} = \begin{bmatrix} \mathbf{R}_0 \otimes \mathbf{d}_h + \mathbf{R}_h \otimes \mathbf{d}_{n-h}, & \frac{1}{\sqrt{2}} \begin{bmatrix} C_0 + C_n \\ S_0 + S_n \end{bmatrix} \otimes \mathbf{d}_s, & \frac{1}{\sqrt{1+C_{vk}}} \begin{bmatrix} C_0 + C_{vk} \\ S_0 + S_{vk} \end{bmatrix} \otimes \mathbf{d}_v \end{bmatrix}$$
$$= \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \mathbf{d}_h + \begin{bmatrix} C_{hk} & -S_{hk} \\ S_{hk} & C_{hk} \end{bmatrix} \otimes \mathbf{d}_{n-h}, & \sqrt{2} \begin{bmatrix} C_0 \\ 0 \end{bmatrix} \otimes \mathbf{d}_s, & \frac{1}{\sqrt{1+C_{vk}}} \begin{bmatrix} 1 + C_{vk} \\ S_{vk} \end{bmatrix} \otimes \mathbf{d}_v \end{bmatrix}, \tag{64}$$

which can be written in a symbolic form as

$$\tilde{\mathbf{A}}^{E_k} = \begin{bmatrix} \varepsilon_1 & -\varepsilon_3 & 0 & \eta_1 \\ \varepsilon_2 & -\varepsilon_4 & \zeta_1 & \eta_2 \\ 0 & 0 & \zeta_2 & \eta_3 \\ \varepsilon_3 & \varepsilon_1 & 0 & \eta_4 \\ \varepsilon_4 & \varepsilon_2 & 0 & \eta_5 \\ 0 & 0 & 0 & \eta_6 \end{bmatrix}, \tag{65}$$

where

$$\begin{aligned} \varepsilon_1 &= C_{(h+v)/2} + S_{h/2} + C_{hk}(-C_{(h-v)/2} + S_{h/2}) \\ \varepsilon_2 &= S_{(h+v)/2} - C_{h/2} + C_{hk}(S_{(h-v)/2} + C_{h/2}) \\ \varepsilon_3 &= S_{hk}(C_{(h-v)/2} - S_{h/2}) \\ \varepsilon_4 &= -S_{hk}(S_{(h-v)/2} + C_{h/2}) \end{aligned}$$

and

$$\begin{array}{lll} \zeta_1 = 2S_v & \zeta_2 = H & \bar{H} = H/[4S_{v/2}(1 - S_{v/2})] \\ \eta_1 = -(1 + C_{vk})S_{v/2} & \eta_2 = (1 + C_{vk})C_{v/2} & \eta_3 = (1 + C_{vk})\bar{H} \\ \eta_4 = -S_{vk}S_{v/2} & \eta_5 = S_{vk}C_{v/2} & \eta_6 = S_{vk}\bar{H}. \end{array}$$

The infinitesimal mechanisms lying in the null-space of $(\tilde{\mathbf{A}}^{E_k})^{\mathrm{T}}$ are

$$\tilde{\mathbf{m}}_1^{E_k} = \begin{bmatrix} \eta_6(\varepsilon_2\varepsilon_3 - \varepsilon_1\varepsilon_4) \\ 0 \\ 0 \\ -\eta_6(\varepsilon_1\varepsilon_2 + \varepsilon_3\varepsilon_4) \\ \eta_6(\varepsilon_1^2 + \varepsilon_3^2) \\ \eta_4(\varepsilon_1\varepsilon_2 + \varepsilon_3\varepsilon_4) - \eta_1(\varepsilon_2\varepsilon_3 - \varepsilon_1\varepsilon_4) - \eta_5(\varepsilon_1^2 + \varepsilon_3^2) \end{bmatrix} \tag{66}$$

and

$$\tilde{\mathbf{m}}_2^{E_k} = \begin{bmatrix} \zeta_2\eta_6(\varepsilon_1\varepsilon_2 + \varepsilon_3\varepsilon_4) \\ -\zeta_2\eta_6(\varepsilon_1^2 + \varepsilon_3^2) \\ \zeta_1\eta_6(\varepsilon_1^2 + \varepsilon_3^2) \\ \zeta_2\eta_6(-\varepsilon_1\varepsilon_4 + \varepsilon_2\varepsilon_3) \\ 0 \\ \zeta_2\eta_4(\varepsilon_1\varepsilon_4 - \varepsilon_2\varepsilon_3) - \zeta_2\eta_1(\varepsilon_1\varepsilon_2 + \varepsilon_3\varepsilon_4) + (\zeta_2\eta_2 - \zeta_1\eta_3)(\varepsilon_1^2 + \varepsilon_3^2) \end{bmatrix}, \tag{67}$$

where

$$\begin{aligned} \varepsilon_2\varepsilon_3 - \varepsilon_1\varepsilon_4 &= 2(S_{v/2} - 1)(-S_{hk}S_h) \\ \varepsilon_1\varepsilon_2 + \varepsilon_3\varepsilon_4 &= -2(S_{v/2} - 1)C_{v/2}(C_{hk} - C_h) \\ \varepsilon_1^2 + \varepsilon_3^2 &= 2(S_{v/2} - 1)[(C_{hk} - C_h)(S_{v/2} + C_h) - S_h^2]. \end{aligned}$$

6. Conclusions

This study shows that symmetry allows simple analytical formulations to be calculated for the study of the mechanics of a whole class of structure. The formulations have been used directly in Part I (Zhang *et al.*, 2008a) to understand the stability of prismatic tensegrity structures. They have also been used in Zhang *et al.* (2008b) to study the stability of dihedral 'star' tensegrity structures.

The methodology in this paper can be applied to any other class of symmetric structure.

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