Multi-stable Star-shaped Tensegrity Structures

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Summary

In this study, we introduce a star-shaped tensegrity structure with 4-fold dihedral symmetry, which is not super stable and can have multiple stable configurations. The transformation is studied between the symmetric initial stable state, and a second unsymmetric state where stability is conferred by contact of struts.

Keywords: *Tensegrity structure; star-shaped tensegrity; stability; multi-stability.*

1.   Introduction

In our previous study [1], we have presented stability conditions for the star-shaped tensegrity structures, which have dihedral symmetry. Figure 1 shows the simplest example of this kind of structure, having three-fold dihedral symmetry. The thick and thin lines in the figure respectively denote struts in compression and cables in tension. By contrast, the structure studied in this paper, on the left side of Figure 2, has four-fold dihedral symmetry.

There are two types of nodes in a star-shaped tensegrity structure: boundary nodes and centre nodes in two parallel planes; and there are three types of members: struts and vertical cables that connect the boundary nodes lying in different horizontal planes, and radial cables that connect the boundary and centre nodes lying in the same horizontal plane.

Super stability [2, 3], where the structure has a unique stable configuration with positive semi-definite geometrical stiffness is usually preferable in the design of tensegrity structures, because they will recover their original configurations after release of any enforced deformations. By using high symmetry of the star-shaped structures, we have proved that their super stability is directly related to the connectivity pattern of the vertical cables: the star-shaped tensegrity structures with
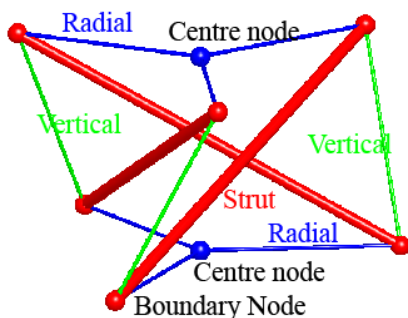

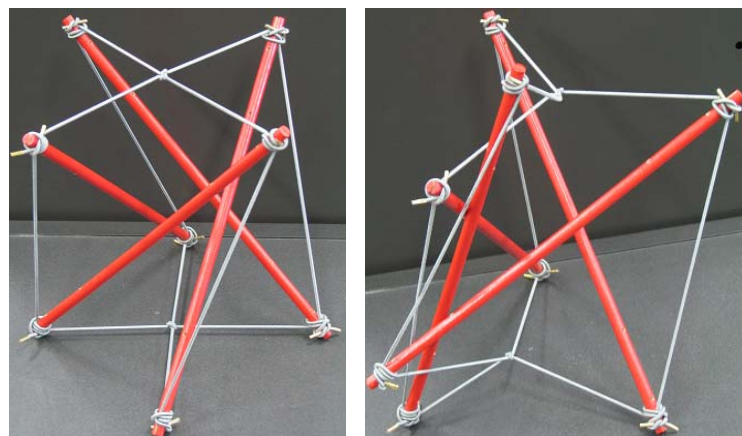*Fig. 1. The simplest 3D start-shaped structure.*

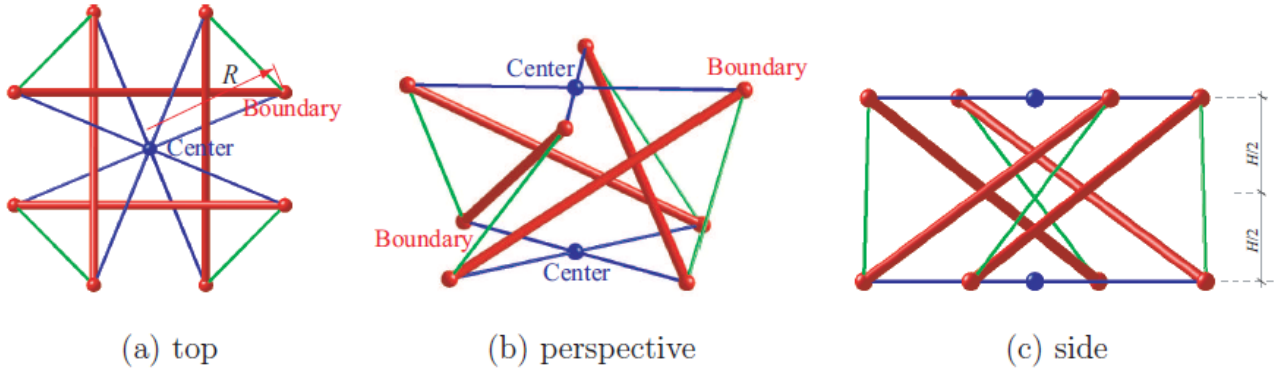*Fig. 2. The two stable configurations.*

Fig. 3. Star-shaped structure with four struts.

dihedral symmetry are super stable if and only if they have odd number of struts, while the struts are as close to each other as possible [1].

In contrast to super stability, the configurations of some structures that are not super stable might be switched to new stable configurations, instead of returning back to their original state, after release of enforced deformations. Such structures are called *multi-stable* structures, because they have more than one stable configuration. This study introduces a multi-stable tensegrity structure – the star-shaped tensegrity structure with 4-fold dihedral symmetry as shown in Figure 2. The configuration on the left side is the original stable one with dihedral symmetry, and that on the right side is another 'stable' configuration with less symmetry.

2.   Configuration and Stability

The structure of interest in the present study, as shown in Figure 3, consists of eight boundary nodes and two centre nodes; there are four struts, eight radial cables and four vertical cables. The original self-equilibrated and stable configuration of the structure is of 4-fold dihedral symmetry. The height of the structure is denoted by H, and the boundary nodes lie at the same radius R from the central axis.
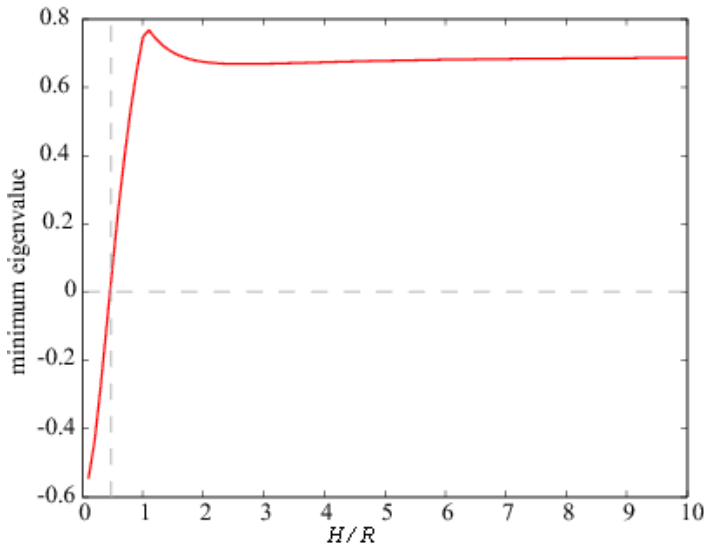

Fig. 4. Minimum (non-dimensionalised) eigenvalue of the tangent stiffness matrix against ratio of height to radius.

At each of the boundary nodes, a strut is in equilibrium with two cables connecting to the same node, all of which must therefore lie in a plane. Thus the out-of-plane displacement of the node must be an infinitesimal mechanism, and accordingly, at least eight infinitesimal mechanisms exist in the structure. In fact, there is another infinitesimal mechanism, as the configuration has been chosen so that the equilibrium matrix [4] is singular, and rank-deficient by one. The corresponding self-equilibrium force mode of the structure is the fully-symmetric mode in which all cables are in tension, and all struts are in compression.

In the unstressed state, the existence of infinitesimal mechanisms implies that the structure cannot be stable in the sense of having positive definite tangent stiffness matrix.

Furthermore, for the structure in the stressed state, we know from our previous study that it is not super stable, because it has even number of struts; however, it turns out the structure is still locally stable with positive definite tangent stiffness, if the level of prestress introduced into the members is not too high so that the negative eigenvalues in the geometrical stiffness matrix do not dominate
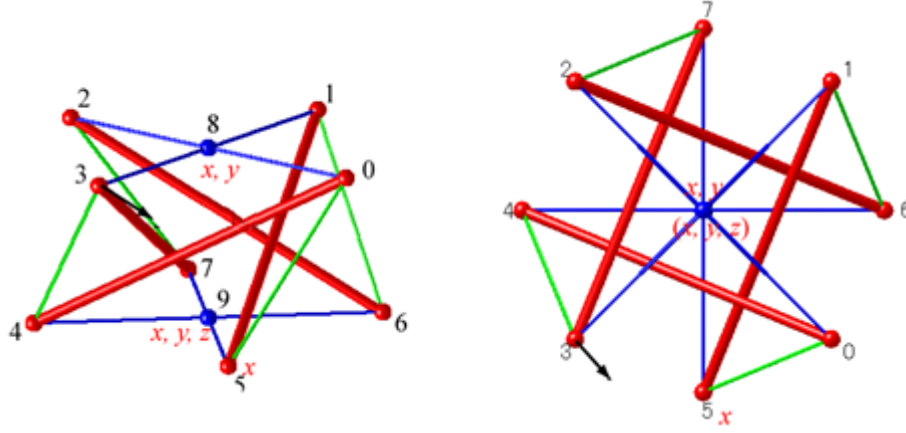
Fig. 5. Analysis model. Statically determinate constraints are applied: node 9 is fully fixed, node 8 is fixed to lie on the z-axis, and node 5 is constrained not to move perpendicular to the 9-5 cable (defining the negative y-axis).

over the positive definiteness of the tangent stiffness matrix, and more importantly, the ratio of height H to radius R of the structure is large enough.

To clarify the last point, we assume that the members of the structure are rigid with infinite axial stiffness, and vary the ratio of height to radius from zero to ten. The minimum eigenvalue of the (simplified) tangent stiffness matrix against the H/R ratio is plotted in Figure 4. Note that the eigenvalues have been non-dimensionalised by force density, defined as the ratio of axial force to member length, of the vertical cables.

The structure is stable if and only if the minimum eigenvalue is positive. Thus, it can be observed from Figure 4 that the structure is stable only when the H/R ratio is larger than 0.5.

Besides the original stable configuration with dihedral symmetry, this structure has another stable configuration with lower symmetry. These two stable configurations are shown in Figure 2, which are physical models made to confirm its distinguished mechanical behaviours. Both of the configurations are stable in the sense that they will recover the original shape after release of small enforced deformations. Moreover, the two stable states can be switched to each other by applying a large enforced deformation – the movement of any node around the principal axis of the high-symmetry configuration.

In the next section, we numerically trace the path, in terms of rotation angle against strain energy, between the two stable configurations.

3.   Multi-stable Path

Figure 2 shows a physical model confirming multi-stable behaviour of the structure, and in this section, we confirm it by numerical investigations.

Rotation is enforced to node 3 about z-axis, and after it is moved counter-clockwise through 45°, it finally arrives at the other stable configuration where there is contact between struts. Displacement control is implemented in the structural analysis, so as to capture the detailed behaviour of the structure during the enforced rotation.

To investigate stability as well as equilibrium of the structure at each displacement (rotation angle) incrementally in the structural analysis, the strain energy stored in the structure is recorded, which is computed by

$$\Pi = \sum_{i=1}^{16} \int_V \frac{1}{2} \sigma_i \varepsilon_i dV = \frac{1}{2} \sum_{i=1}^{16} \frac{\sigma_i^2}{E_i} (A_i l_i) = \frac{1}{2} \sum_{i=1}^{16} \frac{s_i^2 l_i}{A_i E_i} \tag{1}$$

where s_i, l_i, A_i, E_i, $\sigma_i (= s_i / A_i)$ and ε_i are axial force, length, cross-sectional area, Young's modulus, stress and strain of member i, respectively. For simplicity, we assume that cross-sectional areas of

the members are constant, having the values in the initially unstressed state. This should not do much harm to precision of the numerical analysis, because strains in the members are small, though deformations of the structure are large.

The height and radius of the structure are set to $H = R = 1.0$m, and the axial stiffness A_iE_i of the struts and cables are set to 1.0×10^6 N and 1.0×10^2 N, respectively. The force densities of the struts, vertical cables and radial cables in the state of self-equilibrium are -1.0 N/m, 1.0 N/m, and 1.414 N/m, respectively.

The stain energy corresponding to each step of rotated angle is plotted in Figure 6, where the region at the peak of strain energy is amplified for clarity. We can observe from the figure that

1. At the initial position, where the rotated angle is $\theta = 0^\circ$:

   The strain energy is the local minima in the neighbourhood, and therefore, it is in the state of self-equilibrium as well as stability.

2. Between the rotated angle $\theta = 0^\circ$ and that corresponding to the maximum strain energy:

   The strain energy increases associated with the enforced rotation. The structure is equilibrated by the external loads, and is not in the state of self-equilibrium since the gradient of the energy against rotated angle is not equal to zero. Moreover, the structure would return back to the initial configuration $\theta = 0^\circ$ if the external loads are removed.

3. At the rotated angle corresponding to the maximum strain energy:

   The structure is in the state of self-equilibrium since the gradient of strain energy is zero, but it is not stable because the energy is at the peak. It will move back to the initial stable configuration, or move forward to the next 'stable' configuration, depending on the infinitesimal disturbance of external loads.

4. Beyond the rotated angle corresponding to the maximum strain energy:

   The structure is neither stable nor self-equilibrated, and will move forward to the 'stable' configuration with less symmetry as shown on the right side in Figure 2.

5. At the final position, where the rotated angle is $\theta = 45^\circ$ or the constraint line in the figure:

   Although the structure is neither stable nor self-equilibrated from the viewpoint of energy, it is in
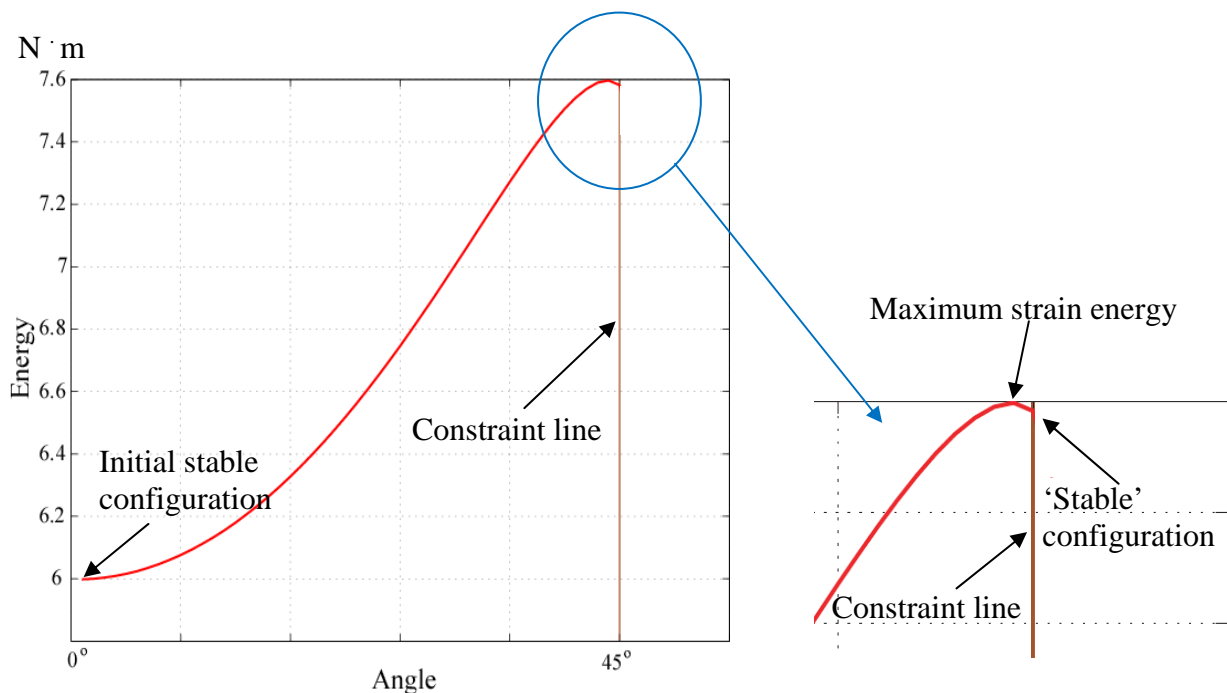

Fig. 6. Strain energy stored in the structure for different rotations of the displaced node around the central axis.

fact equilibrated and stabilized by contact between the compressive members (struts). Therefore, further decrease of the strain energy is prevented.

4. Conclusions

In this study, we have presented a tensegrity structure that can have multiple stable configurations. The multi-stable behaviour of the structure has been confirmed by numerical analyses as well as physical model.

The original stable configuration is of high level of symmetry, in this case 4-fold dihedral symmetry, and the other 'stable' configuration is of less symmetry. Both of these two configurations are stable in the sense that they will recover the original shape after release of small enforced deformations. The original configuration is in the self-equilibrium state with zero gradient of the energy, and furthermore, it is stable with locally minimum energy. By contrast, the alternative 'stable' configuration is in fact equilibrated and stabilized by contact between the struts.

The model can be moved between the two stable states by applying a large enforced deformation – the movement of any node around the principal axis of the high-symmetry configuration.

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5. References

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