Magnetic fields in materials, B and H the truth at last.

Contents

1. How not to teach it.

Electromagnetism has a bad reputation as one of the most difficult subjects in an undergraduate degree. This is not because it is inherently difficult, but because it has been very badly taught by teachers who do not really understand it, and who try to oversimplify the theory so that they can move quickly on to applications. This leads to frustration for the cleverer students who notice that the arguments are far from watertight.
However there are also some good reasons for this situation. One is that electromagnetism is used by physicists, materials scientists and electrical engineers, all of whom have different requirements. This it shares with another difficult subject, thermodynamics. However all thermodynamics can be traced back to the two (or three) fundamental laws, like Euclid's geometry. This is not possible with electromagnetism. The magnetic field can only be derived from a definite integral, or partial differential equations, which are much more difficult concepts. The structure is a bit like a semi-circular arch in which various stones are fitted in, but only when the last is added does the whole thing stand up, particularly when the theory of relativity is incorporated.
Early attempts to avoid this by defining fields experimentally from the force on a current fail in solids such as iron, where the stress in a magnetic field is extremely complex, (1). Forces on bodies in liquids give apparently simple results, but the simplicity is misleading. This macroscopic force approach led to a controversy between Kennelly and Sommerfeld on the definition of magnetic moment. One wanted to define it terms of the torque on a magnet due to $\boldsymbol{H}$ in the fluid, the other due to $\boldsymbol{B}$. (Note that neither made any reference to fields inside the material, these were applied fields in the fluid). Both definitions meant the magnetic moment of an ideal permanent magnet changed when the magnet was immersed in the fluid since they included the magnetisation of the fluid. This is absurd to anyone familiar with the physics of permanent magnets. In fact one theory only works with long thin magnets, the other only with short fat ones (2). The idea of defining fields inside materials from forces on bodies in fluids due to external fields was quite dotty from the start. These forces are a complex combination of hydrostatic and magnetic forces and give no insight into what happens inside solids.
Electrical engineers may be anxious to avoid considering materials on an atomic scale, but this is the only sensible way to do it if we are to understand the fundamentals. The alternative

is just to accept some complex differential equations to be solved, which is certainly not satisfactory for teaching undergraduates in Physics or Engineering, although it is acceptable for Mathematicians.
It is essential to separate out the theory in free space from that in materials, although with a knowledge of the physics of materials, equations can be found which apply to both. Linear magnetic materials are the most difficult since the magnetisation is determined not only by the applied field but also the field due to the magnetisation itself. This large negative feedback in ferromagnetic materials leads to counterintuitive results, such as that the magnetisation of a sphere in an applied field is virtually independent of its permeability, provided this is greater than about 10.
Unfortunately most undergraduate courses persist in beginning with linear materials, although we now have virtually perfect permanent magnets with almost constant magnetisation in large reverse fields which are much easier to understand. Before the recent advances in magnetic materials permanent magnets were very far from ideal (i.e. constant magnetisation), so the subject concentrated on linear materials, with permanent magnets as deviations from linearity It is however essential to introduce the magnetisation as a fundamental field at the very beginning. Once undergraduates get the impression that $\boldsymbol{B}=\mu\boldsymbol{H}$ is a fundamental equation they will never understand the subject. Describing ferromagnets and permanent magnets as non-linear versions of this equation is like describing plastic flow and fracture mechanics as non-linear elasticity. Some equations may work, but there is no understanding.
Another source of confusion is the idea that $\boldsymbol{H}$ in a material is equal to the 'external field', (defined in §4.3). This is the fault of physicists who normally are only interested in long thin samples parallel to an external field, or very low susceptibility materials. These are the only situations where it is true. It makes no sense for electrical engineers, a wire carrying a current does not have an external field but we still need to define $\boldsymbol{H}$ in the material. This confusion led to some erroneous papers in the field of high temperature superconductors where the first single crystals were thin sheets with large demagnetising factors. Sometimes used is the term 'internal field' which is equally misleading, (see below 4.2§ for a discussion of $\boldsymbol{H}$).
There was no need for this confusion. Lorentz had introduced the idea that $\boldsymbol{B}$ was the average of microscopic fields on an atomic scale at the beginning of the twentieth century. This was developed by physics texts in the thirties into a rigorous and comprehensible account , by for example Landau and Lifshitz (3), although their version is a little mathematical. However the mathematics is not essential.
It has taken a long time for these ideas to filter through to text books and undergraduate courses, they certainly had not reached my Physics course in Cambridge in the 1960s. The situation has improved greatly in the last fifty years, although many misconceptions remain. I cannot count how many Ph.D students I have examined who have presented me with pages of magnetisation curves of superconductors, but who were unable to define what was meant by magnetisation.

2. How to teach it.

2.1. The Magnetic field in free space.

This is dealt with in more detail in a section on Maxwell's equations, which will be assumed in the free space version. However we do need to see where the equations come from. The first equations were derived by Biot and Savart using the force between currents and permanent magnets. By regarding the end of a magnet as a pole, like a point charge, they

were able to deduce the magnetic field from a current element. This was a major achievement as the field cannot be measured directly, as can the electrostatic field from a charge, since all currents must flow in a complete circuit.
More accessible experimental results are forces on small current loops which are found to be the same as between two electrostatic dipoles, if we define the moment of the loops by $\boldsymbol{m}=i\boldsymbol{\delta s}$ where i is its current and $\boldsymbol{\delta S}$ the area. From this we can define a field which in free space we can call $\boldsymbol{B}$ or $\boldsymbol{H}$, only the units differ by a factor μ_o .
It is more convenient to use Tesla for both so that the μ_o only appears in force expressions. (The SI unit is A/m, but this is falling out of use). The field is defined similarly to the electrostatic field so that forces and torques on a dipole can expressed in terms of the magnetic field at that dipole due to other sources. In free space we only need one field. Since this is a dipole field div $\boldsymbol{B}$=0, (Gauss theorem) and we can show from either the dipole picture or the Biot-Savart law that curl $\boldsymbol{B}=\mu_o \boldsymbol{J}$ (Ampere's circuital theorem) at frequencies below those for which the displacement current is significant. This essentially means the wavelength must be much larger than the apparatus, (timescales can nevertheless be long), and is the regime considered in most of this article .
Vector analysis shows the two derivations are equivalent. Although the idea of magnetism as due to magnetic poles, like point charges, was an obvious model, it was recognised early on that magnetism might be due to microscopic currents. The concept of magnetic poles remains a useful, if unphysical, technique for solving a number of magnetic problems. However I prefer to use the equivalence between magnetic and electrostatic problems where the poles are replaced by charges, as this is more realistic. The algebra is the same.

2.2. ε_o and μ_o

These fundamental constants have no connection with permeability or permittivity, which are material parameters discussed below (§2.3.4). They are a relic of the time when most materials were linear, atoms were not known about, and physicists wanted to treat fields in materials and free space in a unified theory. This article shows this can be done, but only through an understanding of the atomic nature of materials.
The most fundamental constant is ε_o which is the constant giving the force between charges. Until special relativity we needed another constant giving the force between currents, μ_o. We could then derive the velocity of light, c. However if we compare the force between two lines of charge when they are stationary and when one is moving with a velocity v, relativity tells us the force changes by a factor $1+v^2/c^2$ where c is the velocity of light. This extremely small change, (electron drift velocities are only a few kilometres per hour), can be expressed as the magnetic force between two parallel currents. The large electrostatic forces sum to zero for wires since the proton and electron forces in a solid cancel . The magnetic force is a relativistic correction to the electrostatic force and the fact that we can use this to drive our trains only serves to demonstrate how extremely strong electrostatic forces can be.
To take an example, suppose we take a kilogram of salt and split it into Sodium ions and Chlorine ions. We put the Sodium ions on Cape Wrath at the North West tip of Scotland and the Chlorine ions in Dover about 1000km away. How easy would it be to measure the force between them? The answer is quite easily, it is 2.4 million metric tonnes. ε_o is a very small number.
Since c might be regarded as a more fundamental constant then either ε_o or μ_o, theoretical physicists usually use ε_o and c, rather than ε_o and μ_o, in Maxwell's equations. However this would be inconvenient for engineers and solid state physicists.
None of this affects the definition of units, which has to be done using the most accurate experimental method, rather than the most fundamental constant. We can measure the force

between currents much more accurately than the force between charges so all our electrical units are derived by defining μ_o to be $4\pi x10^{-7}$ henry/m, and the force between currents according to the Biot-Savart law. This determines the ampere and hence the coulomb and all other electrical units.

2.2. Maxwell's Equations in a Magnetic Material, (not Superconductors).

Fields in materials are defined in terms of averages of local fields. The scale on which we average is immaterial provide we are consistent in averaging all fields on the same scale. In a homogeneous material a volume with a large number of atoms is sufficient, while still being much smaller than the dimensions of a macroscopic body. In a multiphase alloy, or a ferromagnetic material with magnetic domains it would need to be on a scale large compared with the microstructure. Whatever it is, the fields will be referred to as 'local' fields, as opposed to fields on an atomic scale which will be called 'microscopic' fields.

The most fundamental and universal field is the flux density ***B*** which is defined as the local average of the microscopic free space field, defined above. It is derived from all currents including electron orbital motion and spin. This applies to all materials, (including superconductors).

In this section we show how an array of dipoles can be averaged to give a macroscopic equivalent current from which the average field can be found. Since this is so central to our understanding two methods will be used. The first uses a specific dipole array which can be generalised if necessary. The second is a reasonably rigorous derivation of Ampères relation in a material.

2.3.1. Magnetisation

Magnetic materials consist of spinning electrons and orbiting electrons each of which behaves as a dipole at distances large compared with the size of an atom. The local magnetisation ***M*** is defined as the sum of the magnetic moments over a small volume, divided by the volume

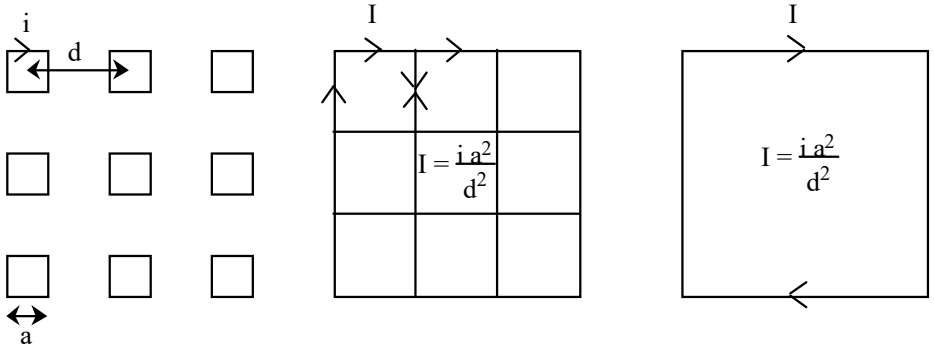


Fig.1. (a) A dipole array. (b) Larger dipoles with the same magnetisation. (c) The Equivalent surface Current.

Figure.1 shows the cross section of a square rod with an array of square dipoles pointing along the axis. They are squares of side a, each carrying a current i, area δs, and with centres separated by d in x, and y directions. A similar array is placed at a distance d below this so that in the z direction the centres of the dipoles coincide, making an array of solenoids. Then $M=ia^2/d^3$. Also we can regard the dipole loops as adjacent turns in a long solenoid with a current density i/d. The flux density in each is $\mu_o i/d$ inside the solenoid and zero outside so the average is :-

$$B = (\mu_o i / d)(a / d)^2 = \mu_o M \qquad (0.1)$$

This is the conventional result for a long permanent magnet in zero external field.

Now let us expand each loop so that it just touches its neighbours. At the same time we reduce the currents so that $i\delta S$ and hence m remain the same. We multiply the edge length by d/a and divide the current by $(d/a)^2$ The dipole moment of each is unaltered so that fields outside the body are unchanged. Also the magnetisation is unchanged. However now all the internal currents cancel leaving only a surface current $i(a/d)^2/a=M$. In other words we can replace the magnetisation by a surface current of M A/m in free space and get the same external field outside the sample.
Fields inside the sample are more problematical, and done properly below (§2.3.2), but it can be seen that the process above keeps the same average field in the sample, as well as the same field outside. Although this seems a little artificial it is not difficult to extend the argument to a more general array. We can change the shape and position of dipoles provided we keep the moment and magnetisation constant.
More general is the case where the magnetisation is not uniform. Suppose that instead of identical currents i, each loop has a current δi greater than its neighbour as we go along the x axis. Now the currents no longer cancel, In fig.1 there will be a net current δ□ along each line in the y direction. The current density is:-

$$J_y = \delta i\left(d^2/a^2\right)/a^2 = -dM_z/dx \qquad (0.2)$$

This is one component of the general equation $\boldsymbol{J}$=curl($\boldsymbol{M}$) which can be used to turn a magnetisation into an 'equivalent' current density. Here equivalent means a current density in free space with the same average field on a local scale at all points inside and outside the sample. Replacing the magnetisation with surface currents equal to the parallel component of a uniform $\boldsymbol{M}$ is a very useful technique, particularly for permanent magnets. However the following is more rigorous and better suited to bulk current densities, leading directly to Maxwell's equations in materials.

*2.3.2 Maxwell's equation for **H** in a material*

Firstly we average the microscopic equation div $\boldsymbol{b}$=0, where $\boldsymbol{b}$ is the microscopic field. To average we integrate each component of $\boldsymbol{b}$ over a small volume and divide by the volume to give the macroscopic equation div $\boldsymbol{B}$=0.
The averaging of the current density is more complex.

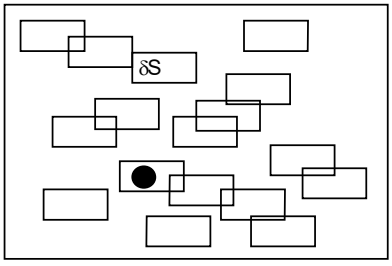


Fig.2. A random array of dipoles in a thickness δl projected onto the xy plane. The dot is the projection of the line element δl perpendicular to the paper. It happens to intersect a current in this plane

On a microscopic scale Ampères theorem states that $\int \boldsymbol{b}.\boldsymbol{dl}=\mu_o I$ so on averaging $\int \boldsymbol{B}.\boldsymbol{d}=\mu_o \bar{I}$. $\bar{I}$ is the mean current, which for the moment we assume to be due to only the local atomic dipoles. Hence we need to work out the average current intersected by a line in a material containing small current loops. Figure 2 shows the projection of the loops of the z component of the magnetisation onto the xy plane for a length δl in the z direction. The line δl is then a point in this plane and the probability of it lying within a particular loop of area δS is $\delta S/A$ where A is the area of the array.

If the loop is intersected there will be a contribution to the line integral of $\boldsymbol{B}$ of i where i is the current in the loop. Otherwise there will be no contribution. Therefore the mean contribution will be the current multiplied by the probability, $i\delta S/A$ or m/A where m is the moment of the loop. The total current for the volume $A\,.\,\delta l$ is therefore $\Sigma m/A$ for this volume. Now the magnetisation M is defined as $\Sigma m/(A\delta l)$, so $\overline{I} = M\,\delta l$. Since $\overline{I}$ is the current due to localised dipoles we must add any macroscopic current I to this mean current to get the total field.
If we divide the currents in this way between atomic dipoles and macroscopic transport currents I, and include components of $\boldsymbol{M}$ in other directions, then Ampères theorem becomes :-

$$\oint \boldsymbol{B}.\boldsymbol{dl} = \mu_o I + \mu_o \oint \boldsymbol{M}.\boldsymbol{dl} \qquad (0.3)$$

We now define $\boldsymbol{H}=\boldsymbol{B}/\mu_o - \boldsymbol{M}$ so that equation (0.3) becomes :-

$$\oint \boldsymbol{H}.\boldsymbol{dl} = I \qquad (0.4)$$

or locally

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} \qquad (0.5)$$

This shows that by taking averages we can write Maxwell's equations in magnetic materials in the same form as in free space, provided we first introduce an extra vector field $\boldsymbol{M}$, the magnetisation. We must also define another field $\boldsymbol{H}$ by $\boldsymbol{H}=\boldsymbol{B}/\mu_o - \boldsymbol{M}$. (Superconductors need a different approach but it can still be done). Only two of the fields $\boldsymbol{B}$,$\boldsymbol{H}$ and $\boldsymbol{M}$ are independent.
It would have been much better to leave out the μ_o so that everything could be measured in Tesla, and the μ_o, would then only appear in expressions for forces and energies, but it is probably too late to change this now.
Purists may feel that introducing a specific physical model is inelegant and unnecessary since the final equations do not depend on it, and it is certainly not how Maxwell did it, but I think it helps to understand what is going on.

2.2.3. The displacement current

Although in general this article is confined to low frequencies, the displacement current in materials is quite easy to incorporate. The displacement current was introduced by Maxwell and was one of his greatest insights since it led to the prediction of electromagnetic waves.
We need to be able to apply Ampères circuital theorem to a circuit with a capacitor. If this contains a dielectric it is clear that the oscillating electrons constitute an AC current which is not a transport current, but which must produce a magnetic field. This current density is $\boldsymbol{J}=q\rho\boldsymbol{v}$ where q is the charge on each atom, ρ the atomic density and $\boldsymbol{v}$ the velocity of the electron. Now the polarisation $\boldsymbol{P}=\rho q\boldsymbol{a}$ where $\boldsymbol{a}$ is the displacement of the charge, so the current density is $\boldsymbol{J}=d\boldsymbol{P}/dt$.
However a capacitor might only contain free space so there must be another contribution to the current if the magnetic field is not to be ambiguous. The way Maxwell used to avoid an ambiguity in Ampères relation, (it is not the only possible one), is to add a term $\varepsilon_o d\boldsymbol{E}/d\boldsymbol{t}$.
Then

$$\nabla \times \boldsymbol{H} = \dot{\boldsymbol{P}} + \varepsilon_o \dot{\boldsymbol{E}} \qquad (0.6)$$

We define $\boldsymbol{D}$ in a material as $\varepsilon_o\boldsymbol{E}+\boldsymbol{P}$, where $\boldsymbol{E}$ is the average of the microscopic electric field, so

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \dot{\boldsymbol{D}} \qquad (0.7)$$

This is the general form of Maxwell's equation and it led directly to the prediction of electromagnetic waves.

Writing (0.7) in this way removes the ambiguity between polarisation currents and transport currents, which is a bit unclear in polymers. Here electrons can travel quite long distances over several seconds, or longer, before falling into a deep hole. Until they stop they are a transport current but when they stop they are polarisation. However the sum of the transport current and displacement current remains the same.

2.3.4. Susceptibility and Permeability

In many materials the magnetisation is proportional to the field on the atom and all average fields are then linearly related so we can write $\boldsymbol{M}=\chi\boldsymbol{H}$ where χ is a material parameter, the susceptibility. Then $B = \mu_o (1+\chi) H = \mu_o \mu_r H$ where $\mu_r = 1+\chi$ and is called the relative permeability.

Note that it is only at this late stage in the development of the subject that the idea of a susceptibility or permeability is introduced. Their usefulness is limited to certain materials, and the results for large permeabilities are often counterintuitive due to demagnetising effects. They do not appear in Maxwell's equations except in the very misleading names for ε_o and μ_o as the permeabilities and permittivity's of free space respectively.

3. Experimental Justification.

There have been a few experiments designed to show that $\boldsymbol{B}$ is the average of the microscopic field. (I am indebted to Dr.Alan Wolsky for these references). The first was by Rasetti (4) who analysed the deflection of cosmic rays in magnetised iron. He found that if the particle is moving fast enough the deflection is that expected from $\boldsymbol{B}$ in the material. This is because a fast particle samples the field in a random way. Slower particles, like muons, can be deflected into regions between atoms with different fields so see a different mean. The theory was described by Wannier (5).

One problem is that the particles must pass through an electron if they are to sample the average field, and the cosmic particles, as well as the neutrons used in later experiments, are considerably larger than the electrons they are passing through. The problem was ignored in fig 2 where it was assumed that the dipoles were current loops which might, or might not, be intersected by a random line. This concept can hardly be applied to electron spins. Similar experimental results were obtained with neutrons (6). A complete theory needs to use the Dirac model of the electron.

More recently muons have been used to explore the internal magnetic fields of materials, in particular superconductors. The muon spin decays at a rate dependent on the magnetic field at the position of the muon, and this decay can be detected since a positron is emitted. It gives good results for the field distribution of flux lines in superconductors, where the field is varying on the scale of about half a micron. However in a magnetised ferromagnetic material a field considerably less than $\boldsymbol{B}$ is detected. This because the muons tend to spend more time at the sides of the atoms where the field is less than the average. However this is not the $\boldsymbol{H}$ field, which is a different average.

4. More Details

In this section we elaborate on some of the questions that the treatment above raises, and dispose of some misconceptions.

4.1. What is Magnetisation?

This term is used to mean different things in different situations and these must not be confused. The magnetisation defined above is a local quantity, averaged over a small

section of the microstructure. In a given material it depends only on the local fields (***H*** or ***B***) and the dependence for any given material can be found in tables of material properties. (Only two of these three vectors are independent). This is the only meaning of ***M*** for which $\boldsymbol{B}=\mu_o(\boldsymbol{H}+\boldsymbol{M})$, which, as seen above, is the definition of ***H***, not ***B***.

Magnetometers measure the total magnetic moment of a body. This can only be defined for a complete body with no currents flowing in or out, (although if end effects are ignored we can also define a magnetisation per unit length for long cylinders). It can be caused by both the atomic scale currents discussed above and also by macroscopic currents such as eddy currents in copper or persistent currents in a superconductor. The total moment is $\boldsymbol{M_o}$, where

$$\boldsymbol{M_o} = \tfrac{1}{2}\mu_o \oint \boldsymbol{r} \times \boldsymbol{j}.dV \qquad (0.8)$$

Here ***j*** is the current density due to both dipoles and macroscopic currents. It gives the dipole component of the field outside the body, which is all that remains at large distances. Quadrupoles and higher harmonics decay more rapidly.

If $\boldsymbol{M_o}$ is divided by the volume we get the mean magnetisation, which is also often shortened to 'Magnetisation'. Provided there are no macroscopic currents this is equal to the integral of ***M*** over the volume of the body so this does not apply to superconductors in the critical state. If there are macroscopic currents their magnetisation has no meaning on a local scale, except that it can be used as the vector potential of the current density, i.e. use curl $\boldsymbol{M}=\boldsymbol{J}$. This can be a useful device in 2D geometries since a scalar M_z can be used to describe the two components of a current in the xy plane to ensure div$\boldsymbol{J}$ =0. This ***M*** is not uniquely defined.

A common misconception is that ***M*** is the difference between the applied field and the mean field in a body. This is only true for long thin cylinders parallel to the applied field. This was the error made in some early measurements of high Tc superconductors which were thin flakes.

Papers by Jan Evetts (7) and Brian Josephson (8) showed in different ways how ***H*** and ***M*** can be defined in superconductors to fit in with the conventional notation of Maxwell's equations. However there are no local dipoles and here ***M*** is the reversible magnetisation as derived from the Abrikosov theory, and so is extremely small in practical Type II superconductors. This means that they can be treated as carrying currents in free space so there is no need to introduce ***H*** in the superconductor since it is always equal to ***B***, (both in Tesla).

4.2. What is H?

The fields ***B*** and ***M*** have simple physical explanations, but attempts to give a similar simple meaning to ***H*** are varied and problematical. The original idea was that the '***H***' was related to the force on currents and the '***B***' to the induced voltage. In free space this is analogous to the difference between inertial and gravitational mass in that there is a very fundamental problem in explaining why they are the same. However the fact that μ_o appears in the expressions both for forces and for induced voltages is inextricably mixed with the generally accepted theory of relativity, so using this concept to separate the fields in materials is not sensible.

As introduced above ***H*** merely combines the physical fields ***B*** and ***M*** in such a way as to make Maxwell's equations more concise and easier to solve. However there are two reasons why ***H*** assumes a greater physical significance than might appear at first sight. One is the practical one that the measurement of magnetic properties is most easily done by magnetising the sample with a coil. If we use a short ferromagnetic sample we need a large field to magnetise it, so instead we use a long thin sample in a long solenoid, or better, a toroidal coil or other complete magnetic circuit. We then measure the flux in the sample with a coil round it and the external field from the current in the magnetising circuit. If there are no transport currents $\boldsymbol{H}_{||}$ is continuous across the surface, in this geometry ***H*** in the material is equal to the ***H*** in the solenoid, i.e. the applied field $\boldsymbol{H_o}$. Also the ***B*** is uniform so we can find it by dividing the flux by the area to get the flux

density. This means that the experimental graph of flux against applied field is the graph of **B** against **H** in the material. The data can then be applied to any point in the material in any geometry, since the relation between $\boldsymbol{B}$ and $\boldsymbol{H}$ in a material is independent of the geometry. Since in ferromagnetic materials $\boldsymbol{B}$ and $\mu_o\boldsymbol{M}$ are very similar in magnitude, a graph of $\boldsymbol{B}$ against $\boldsymbol{M}$ would be more difficult to interpret, although containing the same information in principle. A similar situation occurs in mechanical properties where we measure force against extension in thin cylinders and the use the subsequent stress strain curve in much more complex situations. However to conclude from this, as occasionally stated, that the $\boldsymbol{H}$ causes an $\boldsymbol{M}$ and a $\boldsymbol{B}$ does not make any sense. It is like saying that a stress 'causes' a strain, or a voltage 'causes' a current, when there are many situations where the opposite is true (insofar as either is true). It depends on the reluctance, internal impedance, or stiffness, of the source respectively. It is however true that $\boldsymbol{H}$ gives a qualitative idea of the tendency of a sample to demagnetise. This is discussed in more detail below (§6).

$\boldsymbol{H}$ is sometimes called the 'internal field' but this is not meaningful. The 'internal field' can only mean either the microscopic field on an atomic scale, or its average $\boldsymbol{B}$. Some texts say $\boldsymbol{H}$ is the 'stray field' of a magnet. This is described as the 'field outside the magnet', but since this is usually the field that we want to use, the term 'stray' seems a little uncharitable. It is not a useful concept.

Another wrong idea is that since curl$\boldsymbol{H}=\boldsymbol{J}$, then $\boldsymbol{H}$ is only dependent on transport currents, and is independent of the presence of magnetic materials. This suggests that $\boldsymbol{H}$ can be calculated from currents alone. If this were true we would not need the large number of expensive finite element software programmes which are used to calculate $\boldsymbol{H}$.

In most cases it is important to distinguish between $\boldsymbol{H}$ and the applied field, or external field, usually called $\boldsymbol{H_o}$ or $\boldsymbol{B_o}$. (see §4.3). Since the applied field is normally in free space (we will not consider magnetic fluids), these only differ by a factor μ_o, and in practice Tesla is by far the most convenient unit for both, as well as for $\boldsymbol{M}$. This is reinforced by the coincidental size of the Tesla, which is about the maximum of a ferromagnet, and only a bit less than that from most superconducting magnets. This convention, although not part of the SI system, also means that voltages are directly proportional to field changes and a μ_o only appears in the forces between magnets, and the relation of fields to currents. It has been effectively been adopted by the superconductivity community and also the permanent magnet industry where the magnetisation of NdFeB is quoted in Tesla. They call it the 'remanence' so that it can be quoted in Tesla, but this rather obscures the fact that this is the magnetisation, which is almost constant even in large reverse fields.

The situation was clarified (to some extent) by Josephson (8) who inverted the usual order of derivations by defining $\boldsymbol{H}$ as the gradient of the free energy, F, with respect to the flux density, $\boldsymbol{H}=\nabla_{\boldsymbol{B}}F$, i.e $\delta F=\boldsymbol{H}.\delta\boldsymbol{B}$ defines $\boldsymbol{H}$. It is consistent with an earlier definition as the external field parallel to a surface, although this latter one gives little clue as to fields inside the material. He was then able to derive the equations of the magnetic field in materials in thermodynamic equilibrium, including Type II superconductors. The theory is extended to irreversible materials, but it is unclear how to apply it to permanent magnets.

From this thermodynamic starting point it follows that if $\boldsymbol{H}$ is antiparallel to $\boldsymbol{B}$ the system is thermodynamically unstable , since the free energy can be reduced, although the microscopic mechanism by which the energy is reduced may be obscure. Josephson's treatment is deceptively simple, but is not suitable for undergraduates or electrical engineers, or indeed any but the most dedicated scholars. However it does bring out the fundamental nature of $\boldsymbol{H}$ as an indication of thermodynamic equilibrium in all systems. A non-zero curl $\boldsymbol{H}$ and corresponding $\boldsymbol{J}$ always implies a non-equilibrium situation, even in superconductors.

Therefore notwithstanding the initial remarks above about pictures of $\boldsymbol{H}$, there is indeed a close, but indirect, relation between $\boldsymbol{H}$ and the tendency to demagnetise. However the effect cannot always be explained in terms of reverse fields on dipoles , as discussed below in §6.

4.3 The applied field

The applied field can only be defined for an isolated body. It is the field in the space left by the body if it is removed, and all currents, including all magnetisation currents, except those of the body, are kept constant. It must be done this way round, rather than by adding the body to an existing field, as a magnet near a permeable material experiences an applied field due to its image, which must be preserved when the body is removed. (However in simple geometry the applied field is also the field a long distance from a body). Many problems involving forces and energies are much more easily tackled using the applied field $\boldsymbol{H_o}$ and the total magnetic moment of a body, $\boldsymbol{M_o}$, rather than the local values $\boldsymbol{H}$ and $\boldsymbol{M}$ or $\boldsymbol{B}$ which must be integrated over all space (see the section on energies and forces). This is because we can express the work done on a system in terms of $\boldsymbol{M_o}$ and $\boldsymbol{H_o}$, whereas to find the energies we need to integrate fields over all space which is algebraically intractable. In some geometries the external and internal values are the same, but it is nevertheless extremely important to make clear which are being used.

6. Demagnetising Factors

Demagnetising factors and fields tend to be given a rather cursory treatment in physics courses as there is no fundamental physics involved and the algebra is straightforward. However the result is that in practical applications physicists tend to use results appropriate to paramagnetic materials for ferromagnetic materials, with results that can be wrong by a very large factor. The reason is that in a paramagnetic material when an atom experiences an external field it creates a magnetic dipole, but since the energy is small compared with kT there is not much alignment with the applied field. Hence the resultant field, even from many atomic dipoles, is small compared with the applied field. Susceptibilities are very small and all fields are nearly equal to the applied field.

However in ferromagnetic materials a quantum effect aligns neighbouring spins so there is a large local magnetisation with energies much larger than kT. The spins align with local fields by the movement of domain walls. The result is that as soon as the domain walls move a field is generated which changes the field seen by each atom. There is a large amount of positive or negative feedback which, as we know from electronic circuits, leads to counterintuitive results.

In this section we find local fields, i.e. not microscopic fields but fields over many atoms. We first need to find the fields in a body caused by its magnetisation. We can do this simply for needles ,cylinders, spheres, and slabs.

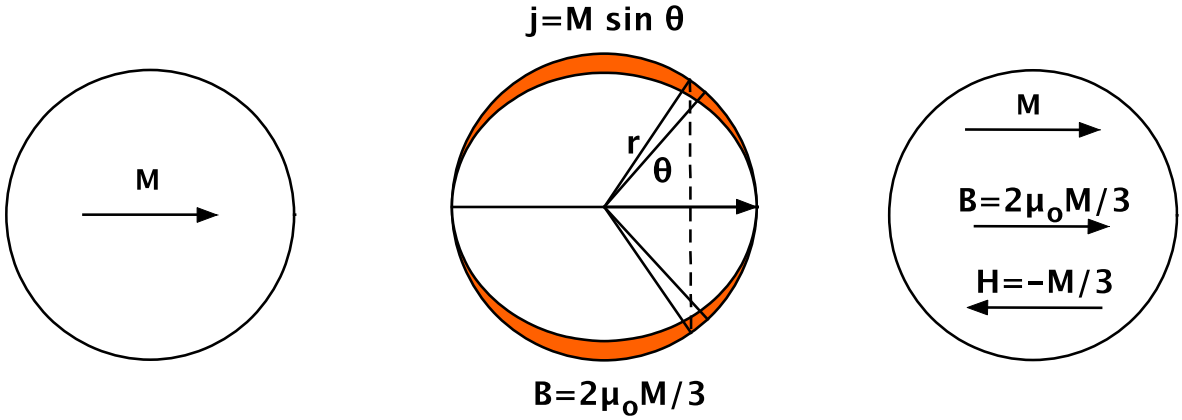


Fig.3a. A Uniformly Magnetised. Sphere b) Equivalent currents. c) B and H.

We consider first a sphere with a magnetisation M along the x axis, fig.3a . We find B in the sphere by replacing it with surface currents in free space, $M\sin\theta$, fig.3b. These are clockwise round the positive x axis. From the Biot-Savart law, at the centre, a current element $jr\,d\theta$ produces a field δB_x in the x direction of magnitude :-

$$\delta B_x = \mu_o M r\delta\theta \sin\theta 2\pi r \sin\theta \sin\theta / 4\pi r^2 \qquad (0.9)$$

Integrating:- $B_x=2\mu_o M/3$. Note that this is positive. The magnetisation creates a field in the material parallel to the magnetisation. This is the value of B at the centre of the sphere so $H=B/\mu_o-M=-1/3M$ so it is H, not the average field B that is opposite to M. These fields are in fact uniform, (proved below). It is only when we find H that we get a field in the opposite direction to the magnetisation.

Since for thermodynamic reasons an H anti-parallel to B tends to demagnetise the sample this is called the demagnetisation field and 1/3 is the demagnetising factor. To this field must be added any external field.

To illustrate the large effect of the sample shape consider a permeable sphere of susceptibility χ in an applied field H_o.

Then $H=-1/3M+H_o$ and $M=\chi H$. Hence

$$M = \frac{\chi H_o}{(1+\chi/3)} \qquad (0.10)$$

For small susceptibility this gives the simple expression, $M=\chi H_o$. However for any ferromagnetic material where $\chi>10$, $M=3H_o$ and $H=H_o/\chi$ and so is very small, while $B=\mu_o M=3\mu_o H_o$. B in the material is three times the applied field (in Tesla) whatever the permeability, so long as it is larger than about ten.

This is an example of the general principle that in most circumstances the magnetisation of a ferromagnetic body in an external field is of the same order as the external field, and independent of the permeability. Only a long thin needle parallel to the field will have a magnetisation of near χH_o.

For this reason if we want to use the magnetic force to attract a permeable body (e.g. in motors and relays) we almost always use a magnetic circuit in which the majority of the reluctance is provided by the air-gap, and the term applied field has little meaning. Again the force will be independent of the permeability, but it is possible to reach the saturation of iron using copper coils in this configuration. Alternatively we use a permanent magnet, which is the favoured choice since the invention of NdFeB magnets.

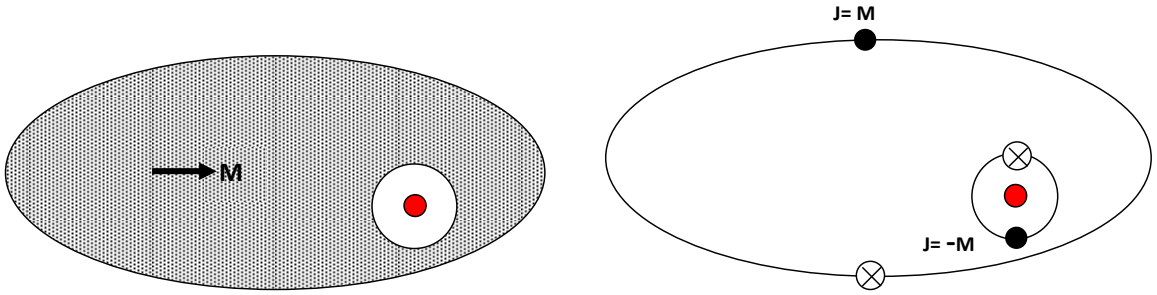
Similar arguments show that the demagnetising factors for transverse cylinders, slabs, and needles are 1/2, 1, and 0 respectively.

5. Microscopic fields.

Because Maxwell's equations are so elegant and complete (at least in the form derived by Heaviside, which is how they are now taught), many texts avoid relating them to any particular atomic picture. This is satisfactory for the more mathematically inclined, but for many students it is very illuminating to see how they work in in simple materials, even if this picture may not be completely general. This was done above with an array of dipoles, although the same equations apply to a domain structure in ferromagnets which is clearly rather different in detail.

The following is the Lorentz theory of dielectrics applied to magnetic materials. Figure. 3 shows an ellipsoid with a demagnetising factor *n,* magnetised in the *x* direction so that from the definition of *n* in the material $H=-nM$ and $B=(1-n)\mu_o M$. We draw a sphere round a particular atom (in red) and divide the field on that atom into the field due to dipoles inside the sphere and those outside.

It can be shown from symmetry that if the atoms are arranged in a simple cubic, hexagonal, or random array, the sum of the fields due to atoms inside a sphere is zero. The field due to atoms outside is that due to surface currents on the outside of the ellipsoid and currents in the opposite direction on the surface of the sphere (fig.3b). To this is added the external field B_o (or H_o).



□□□□□□□3 (a)

(b)

A magnetised ellipsoid with one atom isolated The Equivalent currents

We f□□□□□□□□□□□□□□□□□□□□□□ to connect the susceptibility of a single atom to that of the material. This is the Lorentz theory. □

$$H = H_o - nM \tag{0.11}$$

The field on the atom, B_a, is :-

$$B_a = B_o + \mu_o(1-n)M - \tfrac{2}{3}\mu_o M \tag{0.12}$$

This is a free space field so $B_a=\mu_o H_a$

So

$$B_a = \mu_o\left(\tfrac{1}{3}M + H\right) \tag{0.13}$$

This applies to all materials. We now add the material parameter χ_a, the susceptibility of a single atom, defined by $m=\chi_a H$ where *m* is the moment of an isolated atom and *H* the field applied to it. The atomic density is *N*. Then

$$M = N\chi_a B_a / \mu_o \tag{0.14}$$

So

$$M = \frac{3N\chi_a H}{3 - N\chi_a} \qquad (0.15)$$

This relates the susceptibility (and hence permeability) of a material to that of the atoms of which it is composed. It is however less useful than it might appear. For low susceptibilities the susceptibility of the material is just the simple product of the atomic susceptibility and the atomic density, as might be expected. In this situation *B* and *H* are equal to the external field (if in the same units).

As we increase the susceptibility, or atomic density, when the dilute approximation approaches 3 the material susceptibility diverges, implying ferromagnetism. This is not the origin of ferromagnetism, but it does show the limits of this atomic picture in explaining Maxwell's equations in materials. It only adds a correction to the dilute limit and cannot be used for strongly magnetic material. However it does also illustrate the fact that neighbouring dipoles tend to reinforce an existing magnetisation, rather than demagnetise a sample as suggested by the term demagnetising field.

We now use the Lorentz picture to see how the field on an atom is related to ***B*** and ***H***. We use fig.3 and (0.12) with no external field.

Here are the results for different shapes in zero external field (Table1.)

Shape	**n**	**H**	**B/μ_o**	**B_a/μ_o.**
Slab	1	-M	0	-2/3 M
Sphere	1/3	-1/3M	2/3M	0
Needle	0	0	M	1/3 M

Table 1

We see that if the sample is a sheet normal to the magnetisation (n=1) the field on the atom is in the opposite direction to *B,* but in the same direction as *H* and equal to $2/3\mu_o H$. This clearly tends to demagnetise the sample.

However if the sample is a sphere then B_a=0 and there is no tendency to demagnetise, while if the sample is a long rod parallel to *M* the field is $2/3\mu_o H$ but in the opposite direction to *H* and the same direction as *B*. Thus the field due to other atoms tends to increase *M* so we should have spontaneous magnetisation. This is surprising, and contrary to our experience, and also to general arguments based on reducing magnetic energy by forming closed domains, so where have we gone wrong? One answer is that we cannot apply this picture to the outer layer of atoms, many of which experience a demagnetising field due to their immediate neighbours which is in the opposite direction to *M*. These rotate and the demagnetisation spreads in from the surface until some kind of complex structure with no mean magnetisation is achieved.

(See attached avi file which shows the relaxation of a rectangular array of 2D dipoles on free pivots).

However even if we hold the outer layers stationary, energy arguments tell us the material will still tend to demagnetise. In fact parallel dipoles side by side are inherently unstable on a local scale since they will prefer to turn nose to tail, (like dogs sniffing each other's bottoms. As a child I was given such a pair of magnets with plastic dogs on top for Christmas, but I have not seen them for sale since). The connection of *H* to demagnetisation is therefore a real one, although not necessarily by providing a reverse field, as shown by Josephson. It can involve a collective movement of many dipoles.

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