3E4: Modelling Choice

Lecture 2:

LP:
Spreadsheets
and the
Simplex Method

1

3E4 : Lecture Outline

Lecture 1. Management Science & Optimisation Modelling: Linear Programming

Lecture 2. LP: Spreadsheets and the Simplex Method

Lecture 3. LP: Sensitivity & shadow prices
Reduced cost & shadow price formulae

Lecture 4. Integer LP: branch & bound

Lecture 5. Network flows problems

Lecture 6. Multiobjective LP

Lecture 7 – 8. Introduction to nonlinear programming

2

Returning to the Blue Ridge Hot Tubs Example...

MAX: $350X_1 + 300X_2$ } profit
subject to:
$1X_1 + 1X_2 <= 200$ } pumps
$9X_1 + 6X_2 <= 1566$ } labour
$12X_1 + 16X_2 <= 2880$ } tubing
$X_1, X_2 >= 0$ } nonnegativity

New improved Excel models: OR

3

Goals For Spreadsheet Design

- Communication - A spreadsheet's primary business purpose is that of communicating information to managers.
- Reliability - The output a spreadsheet generates should be correct and consistent.
- Auditability - A manager should be able to retrace the steps followed to generate the different outputs from the model in order to understand the model and verify results.
- Modifiability - A well-designed spreadsheet should be easy to change or enhance in order to meet dynamic user requirements.

4

Some Spreadsheet Design Guidelines

- Organise the data, then build the model around the data.
- Do not embed numeric constants in formulas.
- There is a trade-off between
  - the use of formulas that can be copied
    - reduces implementation time,
    - can sometimes make modifications of the model easier
  - the use of Excel functions like SUMPRODUCT in conjunction with range names
    - easier to validate
    - makes modifications of the model more secure

5

Further Design Guidelines

- Things which are logically related should be physically related.
  - Column/row totals should be close to the columns/rows being totalled.
- The English-reading eye scans left to right, top to bottom.
- Use colour, shading, borders and protection to distinguish elements of your model (data, objective, constraint formulas, changeable parameters)
- Use text boxes and cell notes to document various elements of the model.

6

A Blending Problem: The Agri-Pro Company

- Agri-Pro has received an order for 8,000 pounds of chicken feed to be mixed from the following feeds.

	Percent of Nutrient in			
Nutrient	**Feed 1**	**Feed 2**	**Feed 3**	**Feed 4**
Corn	**30%**	**5%**	**20%**	**10%**
Grain	**10%**	**30%**	**15%**	**10%**
Minerals	**20%**	**15%**	**20%**	**10%**
Cost per pound	**£0.25**	**£0.30**	**£0.32**	**£0.15**

- The order must contain at least 20% corn, 15% grain, and 15% minerals.

7

Agri-Pro Company: Defining the Decision Variables

X_1 = pounds of feed 1 to use in the mix

X_2 = pounds of feed 2 to use in the mix

X_3 = pounds of feed 3 to use in the mix

X_4 = pounds of feed 4 to use in the mix

8

Agri-Pro Company: Defining the Objective Function

Minimize the total cost of filling the order.

MIN: $0.25X_1 + 0.30X_2 + 0.32X_3 + 0.15X_4$

9

Agri-Pro Company: Defining the Constraints

- Produce 8,000 pounds of feed
  $X_1 + X_2 + X_3 + X_4 = 8{,}000$
- Mix consists of at least 20% corn
  $(0.3X_1 + 0.05X_2 + 0.2X_3 + 0.1X_4)/8000 >= 0.2$
- Mix consists of at least 15% grain
  $(0.1X_1 + 0.3X_2 + 0.15X_3 + 0.1X_4)/8000 >= 0.15$
- Mix consists of at least 15% minerals
  $(0.2X_1 + 0.15X_2 + 0.2X_3 + 0.1X_4)/8000 >= 0.15$
- Nonnegativity conditions
  $X_1, X_2, X_3, X_4 >= 0$

10

A Comment About Scaling

- Notice that the coefficient for X_2 in the 'corn' constraint is 0.05/8000 = 0.00000625
- As Solver solves our problem, intermediate calculations must be done that make coefficients larger or smaller.
- Computers have **limited precision**, i.e., use approximations of the actual numbers.
- Such 'scaling' problems may prevent Solver from giving an accurate or reliable solution.

But most problems can be formulated in a way to minimize scaling errors...

Re-Define the Decision Variables

X_1 = ***thousands of pounds*** of feed 1 to use in the mix

X_2 = ***thousands of pounds*** of feed 2 to use in the mix

X_3 = ***thousands of pounds*** of feed 3 to use in the mix

X_4 = ***thousands of pounds*** of feed 4 to use in the mix

Re-Define the Objective Function

Minimize the total cost of filling the order.

MIN: $\mathbf{250}X_1 + \mathbf{300}X_2 + \mathbf{320}X_3 + \mathbf{150}X_4$

13

Re-Define the Constraints

- Produce 8,000 pounds of feed
  $X_1 + X_2 + X_3 + X_4 = \mathbf{8}$
- Mix consists of at least 20% corn
  $(0.3X_1 + 0.5X_2 + 0.2X_3 + 0.1X_4)/\mathbf{8} >= 0.2$
- Mix consists of at least 15% grain
  $(0.1X_1 + 0.3X_2 + 0.15X_3 + 0.1X_4)/\mathbf{8} >= 0.15$
- Mix consists of at least 15% minerals
  $(0.2X_1 + 0.15X_2 + 0.2X_3 + 0.1X_4)/\mathbf{8} >= 0.15$
- Nonnegativity conditions
  $X_1, X_2, X_3, X_4 >= 0$

14

What have we improved?

- Earlier the largest coefficient in the constraints was 8,000 and the smallest is 0.05/8000 = 0.00000625.
- Now the largest coefficient in the constraints is 8 and the smallest is 0.05/8 = 0.00625.
- The problem is now more evenly scaled.

15

The `Use Automatic Scaling` *Option*

The Solver Parameters dialog box has an option:

`Use Automatic Scaling`

(This option button actually has ***no effect*** for LP problems in Excel 5.0 & 7.0 (unless you install the upgraded Solver DLL), but it does work in Excel 8.0.)

This option means your data will be scaled by Solver without you having to do any re-modelling yourself.

16

The **`Assume Linear Model`** *Option*

- The Solver Parameters dialog box has an option labelled
  `Assume Linear Model`
- When you select this option Solver
  - performs some tests to verify that your model is in fact linear
  - uses a specialised LP method
- Linearity tests are not 100% accurate & often fail as a result of poor scaling.
- If Solver tells you a model isn't linear when you know it is, try solving it again. If that doesn't work, try re-scaling your model.

Summary

- Spreadsheets can be used to model and **communicate** optimisation models effectively … with a little **design** effort
- Linear programming software is not perfect but tools like automatic scaling make it much more reliable

18

How does `Solver` *handle LPs?*

The Simplex Method

19

A production example

- Linear programming example

$$\begin{array}{llll} \text{MAX} & 300X_1 + 200X_2 & & \\ \text{Subject to} & 20X_1 + 10X_2 & <= 480 \\ & X_1 + 2X_2 & <= 48 \\ & X_1 & >= 0 \\ & X_2 & >= 0 \end{array}$$

- One method to solve linear programs is the Simplex Method

20

The algebra of the Simplex Method

- Reformulate the problem so that the objective function consists of maximizing a single variable and that the constraints consist only of equations and nonnegativity constraints
- Then manipulate the equations and bear in mind that the variables have to be nonnegative

21

Reformulation

MAX Z

Subject to

$$
\begin{aligned}
20X_1 + 10X_2 + X_3 &= 480 \\
X_1 + 2X_2 + X_4 &= 48 \\
300X_1 + 200X_2 - Z &= 0 \\
X_1, X_2, X_3, X_4 &>= 0
\end{aligned}
$$

The new variables are called slack variables since they measure the slack between left-hand and right-hand side of the corresponding inequalities

22

Reformulation

- System is undetermined (more variables than equations), hence it has lots of solutions (X_1, X_2, X_3, X_4, Z)
- Want to find a solution with
  - X_1, X_2, X_3, X_4 >= 0
  - Largest possible Z-value

23

Basic solutions

- If a system is undetermined, then one may choose certain (so-called basic) variables and express them as functions of the remaining (so-called non-basic) variables
- Choosing X_3, X_4 and Z as basic variables we obtain

$$X_3 = 480 - 20X_1 - 10X_2$$
$$X_4 = 48 - X_1 - 2X_2$$
$$Z = 300X_1 + 200X_2$$

24

Basic solutions

- The particular solution which results from setting all nonbasic variables to zero is called a *basic solution*
- Here: $(X_1, X_2, X_3, X_4, Z) = (0,0,480,48,0)$
- The above form of the system is called a simplex tableau corresponding to the basic variables X_3, X_4 (it is tacitly assumed that Z is always a basic variable)
- A basic solution is called *feasible* if the variables X_i are nonnegative (the Z variable is allowed to be negative)

25

A simplex iteration

- How can we increase Z starting from the basic feasible solution

$$(X_1, X_2, X_3, X_4, Z) = (0,0,480,48,0)?$$

- Last equation implies that Z will automatically increase if X_1 (or X_2) is increased
- That's allowed since increasing X_1 will not violate the nonnegativity condition
- To make Z as large as possible we may want to increase X_1 as far as possible, while X_2 remains fixed to zero

26

A simplex iteration (cont.)

- First equation says that an increase of X_1 leads to a decrease of X_3
- Attention: X_3 is not allowed to drop below zero
- Hence maximal increase of X_1 according to second equation is 48, for otherwise X_4 will become negative
- Since both equations have to be satisfied and X_3 as well as X_4 have to be nonnegative, we deduce that we can change X_1 from $X_1=0$ to $X_1=24$
- The new solution is then

$$(X_1, X_2, X_3, X_4, Z) = (24,0, 0,24,7200)$$

27

Updating the tableau

- In order to repeat the process, we have to change the role of X_1 and X_3 and have to get a simplex tableau for the basic variables X_1, X_4 and Z
- The X_3 equation $X_3 = 480 - 20X_1 - 10X_2$ is equivalent to $X_1 = 24 – 1/20\ X_3 – 1/2\ X_2$
- Plugging this into the remaining two equations yields the new tableau

$$X_1 = 24 - 1/20\ X_3 - 1/2\ X_2$$
$$X_4 = 24 + 1/20\ X_3 - 3/2\ X_2$$
$$Z = 7200 - 15X_3 + 50X_2$$

28

Next iteration

- Can we still improve Z?
- Increasing X_3 doesn't help since Z would decrease
- However, increasing X_2 would increase Z
- If we fix the other nonbasic variables (only X_3) to zero, the nonnegativity conditions for X_1 and X_4 give the bounds
  1. $X_2 <= 48$ (since $X_1 >=0$)
  2. $X_2 <= 16$ (since $X_4 >=0$)
- Hence choose $X_2=16$ and set $X_4=0$ to compensate

29

Next iteration (cont.)

- The X_4-equation is equivalent to

$$X_2 = 16 + 1/30\ X_3 - 2/3\ X_4$$

- Plugging this into the remaining equations we obtain the new tableau

$$X_1 = 16 - 1/15\ X_3 + 1/3\ X_4$$
$$X_2 = 16 + 1/30\ X_3 - 2/3\ X_4$$
$$Z = 8000 - 40/3\ X_3 - 100/3\ X_4$$

- Our method of improving Z stops here since increasing X_3 yields a decrease of Z and so does increasing of X_4

30

Next iteration (cont.)

- Is our current basic solution

  $(X_1, X_2, X_3, X_4, Z) = (16,16,0,0,8000)$

optimal?

- Yes!
- Why?
- Because …

Because …

1. The last tableau describes all feasible solutions for the linear program if we include the condition $X_1, X_2, X_3, X_4 >= 0$
2. The objective value Z does not depend on X_1 or X_2 but only on X_3 and X_4 in the above representation of the feasible solutions
3. Since the coefficients of X_3 and X_4 are negative, Z would be largest possible if X_3 and X_4 were as small as possible
4. The smallest feasible values for X_3 and X_4 are zero
5. The variables achieve these smallest values in the current basic solution
6. Hence Z cannot be increased inside the feasible region and we conclude that the best production plan for the company is to produce 16 units of each product per day

Mathematical basis of SIMPLEX METHOD
Part I

Standard Form of an LP

Let's agree to only consider LPs that look like

MAX $p_1X_1 + p_2X_2 + \ldots + p_nX_n$

Subject to:

$$
\begin{aligned}
a_{11}X_1 + a_{12}X_2 + \ldots + a_{1n}X_n &= b_1 \\
&\vdots \\
a_{k1}X_1 + a_{k2}X_2 + \ldots + a_{kn}X_n &= b_k \\
&\vdots \\
a_{m1}X_1 + a_{m2}X_2 + \ldots + a_{mn}X_n &= b_m \\
X_1, \ldots, X_n \quad &\text{nonnegative.}
\end{aligned}
$$

This is a **Max** LP whose only constraints are linear **equality constraints** and **nonnegative variables**.

Slack Variables

It turns out any LP whose variables are all nonnegative can be put into standard form by introducing (nonnegative) **slack variables**. We know how to put slacks into the Hot Tubs LP:

MAX: $350X_1 + 300X_2$			} profit
subject to:			
$1X_1 + 1X_2 + \mathbf{S_1}$	$=$	200	} pumps
$9X_1 + 6X_2 + \mathbf{S_2}$	$=$	1566	} labour
$12X_1 + 16X_2 + \mathbf{S_3}$	$=$	2880	} tubing
$X_1, X_2, \mathbf{S_1}, \mathbf{S_2}, \mathbf{S_3}$	nonnegative		

35

Vector-Matrix version of LP

Now write the problem in vector notation:

$$\text{Max}\begin{bmatrix}350\\300\\0\\0\\0\end{bmatrix}^{T}\begin{bmatrix}X_1\\X_2\\S_1\\S_2\\S_3\end{bmatrix}\quad\text{subj.to}\begin{bmatrix}1&1&1&0&0\\9&6&0&1&0\\12&16&0&0&1\end{bmatrix}\begin{bmatrix}X_1\\X_2\\S_1\\S_2\\S_3\end{bmatrix}=\begin{bmatrix}200\\1566\\2880\end{bmatrix}$$

$$(X_1, X_2, S_1, S_2, S_3)\quad\text{nonnegative vector}$$

36

Vector-Matrix form continued

It is much easier to write this as

Max $p^T y$ subject to $Ay = b$, y nonnegative

where $y = (X_1, X_2, S_1, S_2, S_3)$, a column vector of decision variables, and the LP data are

$$p^T = \begin{bmatrix} 350 & 300 & 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 9 & 6 & 0 & 1 & 0 \\ 12 & 16 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 200 \\ 1566 \\ 2880 \end{bmatrix}$$

37

Basis *Matrices and a Technical Assumption:* Nondegeneracy

- We'll write our LP in the shorthand standard form: Max $p^T y$ subject to $Ay = b$, y nonnegative.
- Technical assumption: the A matrix, which in general has *m* rows (no. of constraints) and *n* columns (no. of variables), contains *m* columns that form an *m* x *m* **invertible** matrix, B.
  - A is said to be **nondegenerate**
  - B is called a **Basis Matrix** for A
  - the *m* vars. that correspond to the columns of B, chosen from $y_1, y_2, \ldots, y_n$, are called the **Basic Variables**
  - write y_B to denote the vector of *m* basic variables, y_N the vector of $n - m$ **nonbasic** vars.

38

Basic Feasible Solution (BFS)

A **basic feasible solution (BFS)** is a vector y that has a corresponding basis matrix B as follows:

- write $y = (y_B, y_N)$ to denote the basic and nonbasic parts of y, so y_B is in R^m and y_N is in R^{n-m}
- let N = matrix of nonbasic columns of A, so B is in $R^{m\times m}$ and N is in $R^{m\times(n-m)}$, A = [B N] after re-ordering columns of A
- y_B is nonnegative, y_N is zero
- $By_B = b$

Thus a BFS y is nonnegative and $Ay = [B\ \ N]\,(y_B, y_N) = By_B = b$ … **feasible !**

39

Bases for Hot Tubs

When $A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 9 & 6 & 0 & 1 & 0 \\ 12 & 16 & 0 & 0 & 1 \end{bmatrix}$ we have several choices

of bases (basis matrices), including

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \begin{bmatrix} 1 & 1 & 1 \\ 9 & 6 & 0 \\ 12 & 16 & 0 \end{bmatrix}, \text{etc.}$$

which correspond to the basic vector $y_B = (S_1, S_2, S_3)$ in the first case, and $y_B = (X_1, X_2, S_1)$ in the second. So A is nondegenerate.

40

The main point of bases …

LP Basis Theorem. Assume that

- the A matrix is nondegenerate,
- the set of feasible vectors is nonempty, and
- the profit function is bounded above (cannot go to infinity) on feasible set.

Then the LP has a **solution** vector y that is a **BFS**.

Hence if you check the objective function at all BFS's, then the best objective value gives an optimal solution for the LP.

The Simplex Method checks only BFS's, but rarely does it check all of them. (We'll return to this.)

Lecture 2 *3E4 Homework*

1. Use Excel to model and solve Agri-Pro

Make sure to

- Tick **`Assume Linear Model`** and **`Use Automatic Scaling`** in **`Solver`**:**`Options`** window.
- Use appropriate shading to distinguish LP data from changing cells etc.

Lecture 2 *3E4 Homework*

2. Consider Upton Corporation (to follow)
(a) Write down (mathematically) an LP model for this problem
(b) Use Excel to solve it
[Optimal value of LP = £6,061,784]

Make sure to

- Tick **`Assume Linear Model`** and **`Use Automatic Scaling`** in **`Solver`**:**`Options`** window.
- Use appropriate shading to distinguish LP data from changing cells etc.

A Production Planning Problem: The Upton Corporation

- Upton is planning the production of their heavy-duty air compressors for the next 6 months.

	Month					
	1	**2**	**3**	**4**	**5**	**6**
Unit Production Cost	**£240**	**£250**	**£265**	**£285**	**£280**	**£260**
Units Demanded	**1,000**	**4,500**	**6,000**	**5,500**	**3,500**	**4,000**
Maximum Production	**4,000**	**3,500**	**4,000**	**4,500**	**4,000**	**3,500**
Minimum Production	**2,000**	**1,750**	**2,000**	**2,250**	**2,000**	**1,750**

- Beginning inventory = 2,750 units
- Safety stock = 1,000 units
- Unit carrying cost = 1.5% of unit production cost
  - for each month, use average inventory to find carrying cost
- Maximum warehouse capacity = 6,000 units

Lecture 2 *3E4 Homework* *cont.*

3. Blue Ridge Hot Tubs LP:

(a) Find a basis that does NOT correspond to a BFS.

(b) Find a feasible point that is not a BFS.

(c) Show that $(x_1, x_2) = (80,120)$ corresponds to a BFS. Calculate the reduced cost and show that this point is not optimal.

(d) Solve Blue Ridge Hot Tubs by the Simplex Method

45

Lecture 2 *3E4 Homework*

4. For the following LP:

(a) convert to standard form (use <u>max</u>)

(b) solve by the Simplex Method.

$$\begin{array}{lrcl}
\text{MIN} & -5X_1 - 4X_2 - 3X_3 & & \\
\text{Subj. to} & 2X_1 + 3X_2 + X_3 & <= & 5 \\
 & 4X_1 + X_2 + 2X_3 & <= & 11 \\
 & -3X_1 - 4X_2 - 2X_3 & >= & -8 \\
 & X_1, X_2, X_3 & >= & 0
\end{array}$$

[Optimal value of original LP = -13]

46